ERTEL'S POTENTIAL VORTICITY THEOREM IN PHYSICAL OCEANOGRAPHY

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Abstract. Under certain circumstances the equations of motion of a fluid yield theorems that provide powerful aids for comprehension of the character and physics of a wide variety of motions. Perhaps the most powerful of such theorems, especially in geophysical fluid dynamics, are the vorticity theorems that specify how the angular velocity of fluid particles changes with time and position. For large-scale water motions the relevant form of vorticity is what is called potential vorticity, which incorporates inhomogeneities of the constituent elements of seawater. Ertel's theorem specifies the dynamical evolution of potential vorticity. Most other vorticity theorems can be derived from it. For ideal one-component fluids, potential vorticity is materially conserved, that is, it is conserved when a particle of fluid is followed as its location changes, which is a powerful constraint for analyzing fluid motion. Seawater is a two-component fluid with its components being water and salt. Potential vorticity becomes materially conserved for ideal oceanic motions when the thermobaric coefficient (dependent upon the variations with pressure of the thermal expansion and haline contraction coefficients) is assumed to be zero as, for example, for incompressible seawater (incompressibility being a quite valid assumption for many oceanic motions). Approximate forms of potential vorticity are illustrated for standard oceanographic approximations, including a flat Earth with zero, constant, or varying rotation; a spherical Earth; shallow water; stratification; quasi-geostrophy; and others. The concept of potential vorticity naturally defines forms of motion ("vortical" modes) which exhibit nonzero amounts of potential vorticity. The best known forms of "vortical" modes are planetary geostrophic motion (where Coriolis and pressure forces are in approximate balance over the globe), quasi-geostrophic motion (where the forces deviate somewhat from the geostrophic balance), and two-dimensional stratified turbulence. The absence of potential vorticity defines the inertia-gravity mode of motion, that is, gravity waves dependent upon the ocean's stratification. An arbitrary flow can be separated into a part that carries linear motionally induced potential vorticity and a part that does not.

1. INTRODUCTION

The vorticity theorems of Lagrange-Beltrami, Helmholtz, and Ertel and the circulation theorems of Bjerknes and Kelvin feature prominently in any graduate text on geophysical fluid dynamics. *Ertel*'s [1942a] "new hydrodynamic vorticity theorem" plays the most fundamental role for three very different reasons. The first reason is its generality. Other vorticity and circulation theorems, where the latter specify the integration of the velocity along a closed path, can be derived from Ertel's theorem. This fact has been important in the historical evolution of the subject, since it allowed researchers to interrelate theorems based on different assumptions and derive new ones.

The second reason for the importance of Ertel's theorem is the fact that potential vorticity becomes conserved in a variety of circumstances. Ertel's theorem then becomes a conservation law. Conservation laws constrain fluid motions and often simplify their analysis. Two kinds of conservation have to be distinguished: material conservation and global conservation. A fluid property is materially conserved if its value for each fluid particle does not change under fluid motion. A substance is globally conserved if its amount does not change in an appropriately defined fluid volume.

Finally, Ertel's theorem is important because Ertel's potential vorticity equation, or its generalization to nonideal fluids, is the governing equation for an important class of motion. For this class of motion the potential vorticity equation becomes the sole prognostic equation that determines the time evolution of the flow. All other variables can be expressed in terms of the potential vorticity by diagnostic equations. This class of motion includes planetary geostrophic and quasi-geostrophic motions. There is also an important class of motion, inertia-gravity motions, that is characterized by having zero potential vorticity.

Because of its fundamental role, Ertel's theorem yields important insights when applied to oceanic phe-

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nomena. Indeed, most aspects of large-scale oceanography can be understood in terms of potential vorticity and its evolution, as is stressed in textbooks [e.g., *Pedlosky*, 1987] and review articles [e.g., *Rhines*, 1986]. Thus there is considerable current research activity on applications and implications of Ertel's theorem. More surprisingly, there are also new research results about fundamental aspects of Ertel's theorem itself. *Ripa* [1981], *Salmon* [1982], *Henyey* [1982, 1983], and others showed for specific circumstances that the material conservation of potential vorticity is related to a symmetry of the fluid, namely, the invariance of the dynamics under certain particle relabeling transformations. The general result has been given by *Salmon* [1988a]. *Haynes and McIntyre* [1987, 1990] showed that global conservation of potential vorticity holds for nonideal fluids. This review covers these and other fundamental aspects of Ertel's theorem.

The review is divided into four parts (sections 2–5). In section 2 we review basic fluid dynamical aspects. We give a concise derivation of Ertel's theorem (which closely follows Ertel's original derivation), discuss immediate consequences and generalizations, and derive the precise conditions for the conservation of potential vorticity. Section 3 relates the material conservation of potential vorticity to the particle relabeling symmetry. This relation is most easily established in a canonical description of fluid motions, that is, in a description where the equations of motion can be derived from a Lagrangian density using Hamilton's variational principle. Section 4 considers oceanic approximations. It derives and discusses Ertel's theorem for a Boussinesq fluid and lists appropriate forms for the standard oceanographic approximations, for example, the spherical, shallow water, planetary geostrophic, stratified beta plane, quasi-geostrophic, f plane, nonrotating, and two-dimensional turbulence approximations. Section 5 reviews the classes of motion characterized or defined by potential vorticity.

The review aims at answering these questions: (1) What does conservation of potential vorticity reflect? (2) What is the role of materially conserved tracers in the definition of potential vorticity? (3) Which of the approximations made in physical oceanography are essential to the conservation of potential vorticity and which merely simplify the resulting expression?

The review requires the thermodynamic specification of a fluid particle and the equations of motion that govern the dynamical evolution of the flow. It also takes advantage of different flow representations. We switch between Eulerian and Lagrangian descriptions and within the Eulerian description between standard and canonical dependent variables. We also distinguish between prognostic equations that determine the time rate of change of a variable and diagnostic equations that determine the value of a variable. In order not to distract from the main line of reasoning we have added appendices on the thermodynamic specifications of an N-component fluid, the ideal fluid equations in standard Eulerian variables, the Lagrangian description of ideal fluid motions, and the Eulerian forms of Hamilton's principle. These appendices give brief self-contained summaries of these subjects. Similarly, when we discuss oceanic approximation we need basic facts about the thermodynamics of seawater, the equations of motion in a variety of oceanographic approximations, and the Lagrangian density for an incompressible stratified Boussinesq fluid on an f plane. These subjects are also relegated to appendices.

2. BASIC FLUID DYNAMICAL ASPECTS

2.1. Ertel's Theorem

Ertel's theorem is derived from the conservation of mass or continuity equation

$$\frac{D}{Dt} v = v \nabla \cdot \mathbf{u} \tag{1}$$

and the momentum or Euler equation

$$\frac{D}{Dt} \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -v \nabla p - \nabla \Phi \tag{2}$$

which govern the motion of an ideal compressible fluid in a rotating frame of reference. Here $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material derivative, v the specific volume, $\boldsymbol{\Omega}$ the constant angular velocity of the rotating frame, $\mathbf{u}$ the velocity relative to the rotating frame, p the pressure, and Φ an external potential. Owing to the description in a rotating frame, the momentum balance includes the Coriolis acceleration on the left-hand side. The centripetal acceleration is absorbed in the potential Φ (see Appendix C). For oceanographic applications, $\boldsymbol{\Omega}$ will be Earth's rotation rate and Φ the geopotential. The mass and momentum equations do not completely describe the dynamical evolution of a compressible ideal fluid. Additional equations are needed that determine the pressure p. Complete sets of equations are given in Appendix C.

An important property of fluid motion is its vorticity, the curl of the velocity field. In a rotating frame one distinguishes between the relative vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \tag{3a}$$

the "planetary" vorticity

$$2\boldsymbol{\Omega} = \nabla \times \mathbf{U} \tag{3b}$$

and the absolute vorticity

$$\boldsymbol{\omega}^a = 2\boldsymbol{\Omega} + \boldsymbol{\omega} \tag{3c}$$

where $\mathbf{U} = \boldsymbol{\Omega} \times \mathbf{x}$ is the velocity of the rotating frame at position $\mathbf{x}$. The equation of motion for the vorticity is obtained by taking the curl of the momentum balance and has the form

$$\frac{D}{Dt}(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) = (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla \mathbf{u} - (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \nabla \cdot \mathbf{u} - \nabla v \times \nabla p \tag{4}$$

The first term on the right-hand side describes vorticity changes due to vortex stretching and twisting, the second term changes due to volume changes, and the third term changes due to the baroclinicity of the flow. With the help of the continuity equation the vorticity equation can also be written

$$\frac{D}{Dt}[v(\boldsymbol{\omega} + 2\boldsymbol{\Omega})] = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla \mathbf{u} - v(\nabla v \times \nabla p) \tag{5}$$

Changes of the vorticity multiplied by the specific volume are only due to vortex stretching and twisting and to the baroclinicity of the flow.

A fluid property or tracer $\psi(\mathbf{x}, t)$ that satisfies

$$\frac{D}{Dt}\psi = 0 \tag{6}$$

is said to be materially conserved. The property ψ is conserved for each fluid particle individually. With the help of this definition, Ertel's theorem can be stated as follows: If ψ is materially conserved then

$$\frac{D}{Dt} q = vJ(p, v, \psi) \tag{7}$$

where

$$q = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\psi \tag{8}$$

$$J(p, v, \psi) = (\nabla p \times \nabla v) \cdot \nabla\psi \tag{9}$$

The theorem follows directly from the vorticity equation (5) and the material conservation (6) of ψ. The quantity q is called the potential vorticity or Ertel's potential vorticity. It is the projection of the absolute vorticity vector onto the gradient of ψ, multiplied by the specific volume. Ertel's theorem states that material changes of the potential vorticity are given by $J(p, v, \psi)$, the Jacobian of p, v, and ψ. Since ψ can be any materially conserved tracer, potential vorticity represents a class of quantities. Later we will consider specific members of this class by specifying the tracer ψ.

Ertel's potential vorticity theorem (7) and the vorticity theorems (4) and (5) are written in a form that contains the material derivative. Using the continuity equation, one can write these equations also in their flux form. For Ertel's theorem one obtains

$$\frac{\partial}{\partial t}(\rho q) + \nabla \cdot \mathbf{F} = 0 \tag{10}$$

where $\rho = v^{-1}$ is the fluid density and

$$\mathbf{F} = \rho q \mathbf{u} - (\nabla p \times \nabla v)\psi \tag{11}$$

the potential vorticity flux vector. There are no source or sink terms in (10). The local time rate of change of ρq is solely given by the divergence of $\mathbf{F}$. Therefore there exists a "substance" Q, of which q is the amount per unit mass or ρq the amount per unit volume, that is conserved. Since this substance moves with velocity $\mathbf{F}/\rho q$ rather than the fluid velocity $\mathbf{u}$, the integral form of this conservation law is

$$\frac{d}{dt}\iiint_V d^3x \rho q = 0 \tag{12}$$

for any volume V that moves with velocity $\mathbf{F}/\rho q$. We refer to the conservation of the potential vorticity substance Q as expressed in (10) or (12) as the "global" conservation of potential vorticity.

The global conservation of potential vorticity needs to be distinguished from its material conservation, which follows from a trivial corollary to Ertel's theorem. If ψ is materially conserved and if $J(p, v, \psi)$ vanishes identically, then

$$\frac{D}{Dt} q = 0 \tag{13}$$

and potential vorticity is materially conserved. Material conservation of q implies the global conservation of Q but not vice versa. The strength of the corollary is that the existence of one materially conserved tracer ψ implies the existence of another materially conserved tracer, namely q. The corollary also provides the rationale for the adjective "potential." When q is materially conserved, the component of the vector $v\boldsymbol{\omega}^a$ parallel to $\nabla\psi$ increases when $|\nabla\psi|$ decreases. For circumstances where v changes little, absolute vorticity is "released" when adjacent ψ surfaces are spread apart.

Various integral forms of Ertel's theorem and its corollary can be derived. One form has already been given in (12) for a volume moving with velocity $\mathbf{F}/\rho q$. If instead we consider a material volume V, i.e., a volume moving with fluid velocity $\mathbf{u}$, Reynolds's transport theorem implies

$$\frac{d}{dt}\iiint_V d^3\, x\rho q = \iiint_V d^3\, x(\nabla p \times \nabla v) \cdot \nabla\psi \tag{14}$$

Using the vector identities $(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\psi = \nabla[(\boldsymbol{\omega} + 2\boldsymbol{\Omega})\psi]$ and $(\nabla p \times \nabla v) \cdot \nabla\psi = \nabla \cdot [(\nabla p \times \nabla v)\psi]$, application of Gauss's theorem yields

$$\frac{d}{dt}\iint_S d^2x\, \mathbf{n}\cdot(\boldsymbol{\omega}+2\boldsymbol{\Omega})\psi$$

$$=\iint_S d^2x\, \mathbf{n}\cdot(\nabla p\times\nabla v)\psi \tag{15}$$

where **n** is the normal vector of the surface S bounding the volume V. Following *Ertel* [1942b] a form involving a line integral can be obtained by considering a material surface B that divides the fluid into a lower and upper half and a segment A of this surface that is bounded by a simple closed curve δA. Consider a property ψ that is discontinuous across B. Application of (14) to an infinitesimal cylinder with base A leads to (15) with S being replaced by A and ψ being replaced by $\Delta\psi$, the jump of ψ across B. If we choose ψ to be one constant in the lower half and another constant in the upper half, then $\Delta\psi$ drops out and application of Stokes's theorem gives the circulation theorem

$$\frac{d}{dt}\oint_{\delta A} d\mathbf{x}\cdot(\mathbf{u}+\mathbf{U})=\iint_A d^2x\, \mathbf{n}\cdot(\nabla p\times\nabla v) \tag{16}$$

valid for any material circuit δA. This is Bjerknes's circulation theorem [*Bjerknes et al.*, 1933], which is usually written in the form

$$\frac{d}{dt}\oint_{\delta A} d\mathbf{x}\cdot\mathbf{u}+2\Omega\frac{d}{dt}A_\perp$$

$$=\iint_A d^2x\, \mathbf{n}\cdot(\nabla p\times\nabla v) \tag{17}$$

where $A_\perp$ is the projection of A onto the plane perpendicular to $\boldsymbol{\Omega}$. For nonrotating barotropic flows, Bjerknes's circulation theorem reduces to

$$\frac{d}{dt}\oint d\mathbf{x}\cdot\mathbf{u}=0 \tag{18}$$

which is *Kelvin*'s [1869] circulation theorem.

Kinematically, the potential vorticity is related to the rotational part of the velocity field in a generalized Helmholtz decomposition, as shown by *Staquet and Riley* [1989] for a nonrotating system. Let

$$\mathbf{e}_3=\frac{\nabla\psi}{|\nabla\psi|} \tag{19}$$

be the unit vector perpendicular to a ψ surface and $\mathbf{e}_1$ and $\mathbf{e}_2$ two orthogonal unit vectors lying within the ψ surface. The three vectors define an orthonormal coordinate system (ξ_1, ξ_2, ξ_3) with metric coefficients (h_1, h_2, h_3). If ψ is used as the third coordinate, $\xi_3=\xi_3(\psi)$ with $\partial\xi_3/\partial\psi=|\nabla\psi|^{-1}$, then $h_3=1$, and operators within ψ surfaces can be defined. Application of the Helmholtz theorem to the velocity component along the ψ surface results in

$$\mathbf{u}=\mathbf{e}_3\times\nabla\chi+\nabla_i\varphi+u_n\mathbf{e}_3 \tag{20}$$

Here χ is the stream function that describes the rotational, nondivergent part of the velocity component along the ψ surface, φ is the potential that describes the divergent, irrotational part, $u_n=\mathbf{u}_n\cdot\mathbf{e}_3$ is the velocity component perpendicular to the ψ surface, and ∇_i is the gradient within the ψ surface. In this decomposition the potential vorticity is determined by the stream function part

$$q=v\boldsymbol{\omega}\cdot\nabla\psi=v\Delta_i\chi|\nabla\psi| \tag{21}$$

where Δ_i is the Laplacian along the ψ surface. The above arguments can be trivially generalized to rotating systems, by replacing the velocity **u** by the absolute velocity $\mathbf{u}^a=\mathbf{u}+\mathbf{U}$.

2.2. Global Conservation

Ertel's potential vorticity is globally conserved for an ideal fluid. Here we look at two types of generalizations. The first generalization introduces tracers ψ that are not materially conserved. The theorem then takes the form

$$\frac{D}{Dt}[v(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\psi]-v(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\frac{D}{Dt}\psi$$

$$=vJ(p,v,\psi) \tag{22}$$

and allows the derivation of most other vorticity and circulation theorems. For example, by specifying ψ to be the components of the position vector **x**, one retrieves the vorticity equation (5).

The second generalization considers viscous fluids. Let **a** be the viscous force per unit mass or any other additional force per unit mass added to the right-hand side of the momentum equation (2). We then find

$$\frac{D}{Dt}q=vJ(p,v,\psi)+v(\nabla\times\mathbf{a})\cdot\nabla\psi \tag{23}$$

For both generalizations, potential vorticity is not materially conserved, even if $J(p, v, \psi)$ vanishes. However, the new source terms can be written as divergences such that again

$$\frac{\partial}{\partial t}(\rho q)+\nabla\cdot\mathbf{F}=0 \tag{24}$$

with the potential vorticity flux vector

$$\mathbf{F} = \rho q \mathbf{u} - (\nabla p \times \nabla v)\psi - \mathbf{a} \times \nabla \psi - (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \frac{D}{Dt} \psi \tag{25}$$

The potential vorticity substance Q is conserved, even for viscous fluids and nonmaterially conserved tracers ψ. Furthermore, this substance cannot cross ψ surfaces since the normal component of the velocity $\mathbf{F}/\rho q$ equals the normal velocity of the ψ surface:

$$\frac{\mathbf{F} \cdot \nabla \psi}{\rho q} = -\frac{\partial \psi}{\partial t} \tag{26}$$

These fundamental results have apparently only been realized recently by *Haynes and McIntyre* [1987, 1990]. Equations (24) and (26) imply that the potential vorticity substance can only move along ψ surfaces and is created or destroyed only where these surfaces intersect boundaries. If a fluid volume is solely bounded by ψ surfaces, then the potential vorticity substance in that volume does not change. It can only be diluted or concentrated by changes in the mass or the volume. In the atmosphere, isentropic surfaces are usually chosen as ψ surfaces. In the middle and upper atmosphere, layers can be found that are bounded solely by two isentropic surfaces, and the global conservation of potential vorticity becomes a constraint for real fluid motion that is presently exploited by atmospheric scientists [e.g., *McIntyre*, 1990]. In the ocean, where isopycnal surfaces are usually chosen as ψ surfaces, global conservation is less of a constraint since isopycnal surfaces usually intersect the ocean surface or the bottom.

The global conservation of potential vorticity and its confinement to ψ surfaces are solely a consequence of the definition of the potential vorticity as

$$\rho q = (\nabla \times \mathbf{u}^a) \cdot \nabla \psi \tag{27}$$

where $\mathbf{u}^a = \mathbf{u} + \mathbf{U}$. By applying vector identities we find that ρq can be written as a divergence:

$$\rho q = \nabla \cdot [(\nabla \times \mathbf{u}^a)\psi] = \nabla \cdot [\mathbf{u}^a \times \nabla \psi] \tag{28}$$

Therefore

$$\frac{\partial}{\partial t} (\rho q) = \nabla \cdot \left[(\nabla \times \mathbf{u}^a) \frac{\partial \psi}{\partial t} + \frac{\partial \mathbf{u}^a}{\partial t} \times \nabla \psi \right] \tag{29}$$

which is (24) with flux vector

$$\mathbf{F} = -(\nabla \times \mathbf{u}^a) \frac{\partial \psi}{\partial t} - \frac{\partial \mathbf{u}^a}{\partial t} \times \nabla \psi \tag{30}$$

This flux vector also satisfies (26). Equation (28) shows that the amount of the potential vorticity substance in a volume V can either be determined by integrating ρq over the volume or by integrating $(\nabla \times \mathbf{u}^a)\psi$ or $\mathbf{u}^a \times \nabla \psi$ over the surface bounding the volume.

2.3. Material Conservation

Material conservation of potential vorticity is inevitably destroyed if viscous, conductive, and diffusive processes are introduced. Thus potential vorticity is never materially conserved for a real fluid. Even for an ideal fluid the corollary to Ertel's theorem shows that $J(p, v, \psi)$ must vanish for potential vorticity to be materially conserved. This condition is analyzed here.

The Jacobian $J(p, v, \psi)$ vanishes if v is a function of p and ψ. This is a constraint on the equation of state of the fluid. Consider a two-component fluid. Its equation of state is of the general form

$$v = v(p, \eta, c) \tag{31}$$

where η is the specific entropy and c the concentration of the second component (see Appendix A). For ideal fluid motions the specific entropy η and the concentration c are materially conserved. The Jacobian $J(\rho, v, \psi)$ therefore vanishes if a tracer

$$\psi = \psi(\eta, c) \tag{32}$$

can be found such that the equation of state reduces to

$$v = v(p, \psi) \tag{33}$$

Note that ψ is a thermodynamic tracer, whereas the potential vorticity q is a dynamic tracer. A thermodynamic tracer depends only on thermodynamic variables and is a property of the fluid. A dynamic tracer depends also on the fluid velocity and is a property of the flow. The corollary to Ertel's theorem states that for every materially conserved thermodynamic tracer ψ that satisfies (33) there exists a materially conserved dynamic tracer, namely q. The inference of a property of the flow from a property of the fluid constitutes the significance of the corollary.

For which equations of state can a ψ be found that satisfies (33)? Consider compressible fluids ($\kappa \neq 0$) first. For a one-component fluid, $v = v(p, \eta)$ and one can trivially choose $\psi = \eta$ and

$$q = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla \eta \tag{34}$$

becomes materially conserved. If the fluid is furthermore homentropic ($\nabla \eta = 0$) then $\nabla p \times \nabla v = 0$ and hence $J(p, v, \psi) = 0$ for any ψ. Potential vorticity is materially conserved for any tracer ψ. However, there is no obvious candidate for such a tracer.

For a two-component or multicomponent fluid, (33) requires that the fluid essentially behave as a one-component fluid, i.e., it can be brought into the one-component form by a transformation of variables. The condition $J(p, v, \psi) = 0$ implies

$$\left(\frac{\partial \psi}{\partial c} \frac{\partial v}{\partial \eta} - \frac{\partial \psi}{\partial \eta} \frac{\partial v}{\partial c} \right) (\nabla p \times \nabla \eta) \cdot \nabla c = 0 \tag{35}$$

after substitution of $\psi = \psi(\eta, c)$ and $v = v(p, \eta, c)$. Either the term in parentheses or the triple product

TABLE 1. Summary of the Materially Conserved Potential Vorticities for Various Equations of State

Equation of State	*Form of Materially Conserved Potential Vorticity*
Two-component fluid ($v = v(p, \eta, c)$)	none
Two-component fluid with nonturning isopycnals ($v = v(p, v_*)$)	$q = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla v_*$
One-component fluid ($v = v(p, \eta)$)	$q = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla \eta$
Homentropic fluid ($v = v(p; \eta)$, $\nabla \eta = 0$)	?
Incompressible fluid	$q = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla v$
Homogeneous fluid ($\nabla v = 0$)	?

must vanish. If the triple product vanishes, $p = p(\eta, c)$ and hence $v = v(\eta, c)$. The fluid is incompressible, the case not considered here. Therefore the term in parentheses must vanish. Since v is a function of p whereas ψ is not, this can only happen if the slope

$$\gamma = \frac{\left(\dfrac{\partial v}{\partial c}\right)_{p,\eta}}{\left(\dfrac{\partial v}{\partial \eta}\right)_{p,c}} \tag{36}$$

of the isopycnal in the (η, c) plane does not change with pressure. Such a fluid behaves as a one-component fluid since the specific volume is determined by the pressure p and the specific volume $v_* = v(p_*, \eta, c)$ at an arbitrary reference pressure p_*, i.e., $v = v(p, v_*)$. The choice $\psi = v_*$ leads to a materially conserved potential vorticity. This case is more fully discussed in the oceanographic sections since it is of particular importance there.

Finally, consider the incompressible fluid limit $\kappa = 0$. In this case

$$\frac{D}{Dt} v = 0 \tag{37}$$

The specific volume is materially conserved and $\psi = v$ leads to a materially conserved potential vorticity. Again, there is no obvious candidate for ψ if the fluid is homogeneous, $\nabla v = 0$.

The material conservation of potential vorticity implies the circulation theorem

$$\frac{d}{dt} \oint_{\delta A} d\mathbf{x} \cdot (\mathbf{u} + \mathbf{U}) = 0 \tag{38}$$

where δA is a material circuit on an isentropic (isopycnal) surface for a one-component (incompressible) fluid or any material circuit for a homentropic or homogeneous fluid.

Table 1 summarizes the essence of this section. It is the equation of state that determines whether or not there exists a materially conserved potential vorticity and what form it takes. The results in Table 1 can be related to the number of modes of motion a fluid can support. A two-component system is described by six prognostic equations, e.g., for the velocity vector $\mathbf{u}$, the pressure p, the specific entropy η, and the concentration c (Appendix C). Such a fluid can therefore support six modes of motion. In the linear limit, two of the modes correspond to acoustic waves, two to gravity waves, one to a vorticity wave, and one to a chemical wave. For a one-component fluid the number of prognostic variables is reduced to five. The chemical mode is eliminated. An incompressible fluid is governed by three prognostic equations. The two acoustic modes and the chemical mode are eliminated. Once the chemical mode is eliminated, there exists a materially conserved potential vorticity. The presence or the absence of the acoustic modes does not affect the existence of a materially conserved potential vorticity but only affects its form.

3. PARTICLE-RELABELING SYMMETRY

3.1. Lagrangian Formulation

The analysis of the previous section is unsatisfactory since it neither reveals the underlying cause for the material conservation of potential vorticity nor offers any explicit expressions for homentropic and homogeneous fluids. These issues become resolved in a Lagrangian description of the fluid motion.

In the Lagrangian frame the fluid motion is described by the position $\mathbf{x}$ of a fluid particle as a function of its label $\mathbf{s}$ and time τ, i.e., by $\mathbf{x} = \mathbf{x}(\mathbf{s}, \tau)$. Often the initial or equilibrium position $\mathbf{r}$ is used as a fluid label. To avoid ambiguities, time is denoted by τ in the Lagrangian description and by t in the Eulerian description. Similarly, the gradient in label space is denoted by $\tilde{\nabla}$ and the gradient in physical space by ∇. The Lagrangian description is equivalent to the Eulerian one. A more extensive discussion of the Lagrangian description can be found in Appendix D.

An important property of the Lagrangian description is that the momentum equation for ideal fluids can be derived from a Lagrangian density using Hamilton's variational principle. The principle was first given by *Herivel* [1955] for an incompressible fluid and later by *Serrin* [1959] and *Eckart* [1960] for a compressible flow. For a compressible fluid in a rotating frame of reference the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\,\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \dot{\mathbf{x}} \cdot \mathbf{U} - e - \Phi \qquad (39)$$

where $\dot{\mathbf{x}} = (\partial/\partial\tau)\mathbf{x}(\mathbf{s}, \tau)$ is the fluid velocity and e the specific internal energy. The Lagrangian density is the kinetic energy density minus the internal and potential energy density. The expression (39) is a density in label space. For convenience the label $\mathbf{s}$ is chosen such that the initial density $\rho_0(\mathbf{r})$ is uniform in label space, i.e., $ds_1 ds_2 ds_3 = \rho_0(r) dr_1 dr_2 dr_3$. We will adopt this choice of label unless stated otherwise.

For a two-component system, $e = e(v, \eta, c)$. In the Lagrangian description the continuity equation takes the form

$$v(\mathbf{s}, \tau) = \frac{\partial(\mathbf{x})}{\partial(\mathbf{s})} = K \qquad (40)$$

The specific entropy η and the concentration c are materially conserved for ideal fluid motion, and their values are given by their initial values

$$\eta(\mathbf{s}, \tau) = \eta_0(\mathbf{s}) \qquad c(\mathbf{s}, \tau) = c_0(\mathbf{s}) \qquad (41)$$

Thus $e = e(K, \eta_0, c_0)$.

An important theorem that governs Lagrangian systems is due to *Noether* [1918] and reviewed by *Hill* [1951]. It states: If the Lagrangian density is invariant under infinitesimal transformations $\delta\tau$, $\delta\mathbf{s}$, or $\delta\mathbf{x}$, then there exists a "global" conservation law of the form

$$\frac{\partial}{\partial t}\left(\mathcal{L}\delta\tau + \frac{\delta\mathcal{L}}{\delta x_{i,\tau}}\,\Delta x_i\right) + \frac{\partial}{\partial s_j}\left(\mathcal{L}\delta s_j + \frac{\delta\mathcal{L}}{\delta x_{i,j}}\,\Delta x_i\right) = 0 \qquad (42)$$

or

$$\frac{d}{d\tau}\iiint d^3s\left(\mathcal{L}\delta\tau + \frac{\delta\mathcal{L}}{\delta x_{i,\tau}}\,\Delta x_i\right) = 0 \qquad (43)$$

for suitable volumes or boundary conditions. Here

$$\Delta x_i = \delta x_i - x_{i,\tau}\delta\tau - x_{i,j}\delta s_j \qquad (44)$$

is the variation of $\mathbf{x}$ at fixed $\mathbf{s}$ and τ. Index notation is used, and the functional derivative is denoted by δ. Noether's theorem relates the conservation of momentum to the invariance of the Lagrangian density under a uniform translation in physical space and the conservation of energy to the invariance under a uniform translation in time.

Consider the Lagrangian density (39) for an ideal compressible fluid. A particle-relabeling transformation

$$\mathbf{s} \rightarrow \mathbf{s}' = \mathbf{s} + \delta\mathbf{s} \qquad (45)$$

only affects K, η_0, and c_0 in the internal energy term. Hence, if a transformation $\delta\mathbf{s}$ can be found that does not change the internal energy, then the Lagrangian density is invariant under this transformation and Noether's theorem applies. Such a transformation can indeed be found for a one-component fluid for which $e = e(K, \eta_0)$. The condition $\delta K = 0$ implies $\tilde{\nabla} \cdot \delta\mathbf{s} = 0$, which is satisfied by $\delta\mathbf{s} = \tilde{\nabla} \times \delta\mathbf{A}(\mathbf{s})$ where $\delta\mathbf{A}(\mathbf{s})$ is an arbitrary infinitesimal vector field. The condition $\delta\eta_0 = 0$ requires $\delta\mathbf{s} \cdot \tilde{\nabla}\eta_0 = 0$, which is satisfied if $\delta\mathbf{A} = \delta a(\mathbf{s})\tilde{\nabla}\eta_0$ where $\delta a(\mathbf{s})$ is an arbitrary infinitesimal scalar field. The internal energy and the Lagrangian density are hence invariant under the infinitesimal transformation

$$\delta\mathbf{s} = \tilde{\nabla} \times (\delta a\tilde{\nabla}\eta_0) \qquad (46)$$

This invariance represents the fact that the thermodynamic state of a one-component fluid is completely specified by two scalar variables, say, v and η, whereas the label represents a three-dimensional manifold, s_1, s_2, and s_3. The one-dimensional relabeling transformation (46) can be applied without changing the thermodynamic state of the fluid. The thermodynamic state of a genuine two-component fluid is characterized by three scalar variables, say, v, η, and c, and no such relabeling transformation exists.

The conservation law associated with the relabeling symmetry can be found by substituting (46) into the integral form (43) of Noether's theorem. One obtains

$$\frac{d}{d\tau}\iiint d^3s\;\mathbf{B} \cdot [\tilde{\nabla} \times (\delta a\tilde{\nabla}\eta_0)] = 0 \qquad (47)$$

where the vector $\mathbf{B}$ has the components

$$B_i = \frac{\delta\mathcal{L}}{\delta x_{j,\tau}}\,x_{j,i} = (\dot{x}_j + U_j)\,\frac{\partial x_j}{\partial s_i} \qquad (48)$$

and is discussed in the next section. Integration by parts yields

$$\frac{d}{d\tau}\iiint d^3s\;(\tilde{\nabla} \times \mathbf{B}) \cdot \tilde{\nabla}\eta_0\delta a = 0 \qquad (49)$$

This is a global conservation law. However, since $\delta a(\mathbf{s})$ is an arbitrary function of $\mathbf{s}$ we can choose it to be a delta function and obtain the material conservation law

$$\frac{\partial}{\partial\tau}\,\tilde{q} = 0 \qquad (50)$$

where

$$\tilde{q} = (\tilde{\nabla} \times \mathbf{B}) \cdot \tilde{\nabla}\eta_0 \qquad (51)$$

Transforming from label to physical space, we find

$$\tilde{q} = v[\nabla \times (\mathbf{u} + \mathbf{U})] \cdot \nabla\eta \qquad (52)$$

i.e., $\tilde{q}$ is the potential vorticity q. The material conservation of potential vorticity for an ideal one-component fluid reflects the fact that a one-dimensional particle-relabeling transformation can be carried out without changing the thermodynamic state of the fluid. This connection between potential vorticity conservation and particle-relabeling symmetry has been recognized by *Bretherton* [1970], *Ripa* [1981], *Salmon* [1982], *Henyey* [1982, 1983], and others for specific circumstances. The above general derivation was first given by *Salmon* [1988a].

For a homentropic fluid, $e = e(K)$ and the Lagrangian density is invariant under the transformation

$$\delta \mathbf{s} = \tilde{\nabla} \times \delta \mathbf{A} \tag{53}$$

where $\delta\mathbf{A}(\mathbf{s})$ is an arbitrary infinitesimal function of $\mathbf{s}$. Noether's theorem then implies

$$\frac{d}{d\tau} \iiint d^3 s (\tilde{\nabla} \times \mathbf{B}) \cdot \delta \mathbf{A} = 0 \tag{54}$$

or

$$\frac{d}{d\tau} \mathbf{\Pi} = 0 \tag{55}$$

where

$$\mathbf{\Pi} = \tilde{\nabla} \times \mathbf{B} \tag{56}$$

Now the vector $\mathbf{\Pi}$ is materially conserved. Transforming to physical space, we find

$$\mathbf{\Pi} = (\tilde{\nabla} \times \mathbf{B}) \cdot \tilde{\nabla} \mathbf{s} = v[\nabla \times (\mathbf{u} + \mathbf{U})] \cdot \nabla \mathbf{s} \tag{57}$$

which cannot be expressed in standard Eulerian variables because it involves the label $\mathbf{s}$. As pointed out by *Salmon* [1988a], the conservation law (55) implies not only Kelvin's circulation theorem but also *Moffatt*'s [1969] helicity theorem.

For an incompressible fluid the pressure is no longer determined "thermodynamically" by the specific internal energy but "dynamically" by the incompressibility constraint,

$$K = v_0(\mathbf{s}) \tag{58}$$

This constraint can be added to the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \dot{\mathbf{x}} \cdot \mathbf{U} - \Phi + \lambda[K - v_0(\mathbf{s})] \tag{59}$$

where the Lagrange multiplier λ must be identified with the fluid pressure p. The Lagrangian density (59) is invariant under relabeling transformations that do not change K and $v_0(\mathbf{s})$. Arguments similar to the one-component case lead to

$$\frac{\partial}{\partial \tau} [(\tilde{\nabla} \times \mathbf{B}) \cdot \tilde{\nabla} v_0] = 0 \tag{60}$$

and to

$$\frac{\partial}{\partial \tau} (\tilde{\nabla} \times \mathbf{B}) = 0 \tag{61}$$

for the homogeneous case.

Noether's theorem provides the underlying reason for the material conservation of potential vorticity. It is conserved when fluid particles can be relabeled without changing the thermodynamic state of the fluid. Since Noether's theorem requires infinitesimal transformations, potential vorticity conservation does not exist in discrete particle physics. For a one-component fluid whose thermodynamic state is determined by v and η, there exist one-dimensional relabeling transformations leading to the conservation of a scalar. For a homentropic fluid whose thermodynamic state is completely determined by v, there exist two-dimensional relabeling transformations leading to the conservation of two scalars or, equivalently, to the conservation of the curl of a vector. The relabeling transformations are local transformations, that is, they depend on $\mathbf{s}$, and lead to local conservation laws, that is, to quantities that are materially conserved. In noncanonical Hamiltonian formulations the conservation of potential vorticity is related to the existence of Casimir functionals. This aspect is also discussed by *Salmon* [1988a].

3.2. Pseudomomentum

The vector $\mathbf{B}$ that appears in the Lagrangian forms of potential vorticity conservation is a pseudomomentum vector as can be seen by inverting (48), which yields

$$\dot{\mathbf{x}} + \mathbf{U} = B_1 \nabla s_1 + B_2 \nabla s_2 + B_3 \nabla s_3 \tag{62}$$

The components of $\mathbf{B}$ are the components of the absolute velocity vector in a basis given by the three curvilinear vectors ∇s_i $(i = 1, 2, 3)$. In contrast, the momentum vector $\mathbf{P}$ has components

$$P_i = \frac{\delta \mathcal{L}}{\delta x_{i,\tau}} = u_i + U_i \tag{63}$$

that satisfy

$$\dot{\mathbf{x}} + \mathbf{U} = P_1 \nabla x_1 + P_2 \nabla x_2 + P_3 \nabla x_3 \tag{64}$$

and are the components of the absolute velocity vector in a basis given by the three unit vectors ∇x_i $(i = 1, 2, 3)$. Because of our specific choice of label there appears no density in the definitions for momentum and pseudomomentum.

Pseudomomentum is globally conserved under circumstances that are determined by particle-relabeling symmetries. Consider a homentropic fluid, $e = e(K)$. The Lagrangian density does not depend on the label coordinates and is invariant under a global or uniform infinitesimal translation

$$\delta \mathbf{s} = \text{const} \tag{65}$$

in label space. The differential form (42) of Noether's theorem then implies

$$\frac{\partial}{\partial t} B_i + \frac{\partial}{\partial s_j} F_{ij} = 0 \qquad i = 1, 2, 3 \tag{66}$$

with a certain flux tensor F_{ij}. All components of the pseudomomentum vector are globally conserved. By taking the curl of (66) in label space, we recover the material conservation of potential vorticity (55).

For a one-component fluid, $e = e(K, \eta_0)$. If one chooses η_0 as one of the labeling coordinates, say, $s_3 = \eta_0$, then the Lagrangian density is independent of s_1 and s_2, leading to the global conservation of the pseudomomentum components B_1 and B_2 and the material conservation of the potential vorticity $q = (\tilde{\nabla} \times \mathbf{B})_3 = (\tilde{\nabla} \times \mathbf{B}) \cdot \tilde{\nabla}\eta_0$. For a two-component system, $e = e(K, \eta_0, c_0)$ and the choice $s_3 = \eta_0$ and $s_2 = c_0$ leads to the global conservation of B_1 but to no material conservation law.

The global conservation of the pseudomomentum vector or its components follows from the invariance of the Lagrangian density under a global particle-relabeling transformation, such as (65). The material conservation of potential vorticity follows from the invariance under local particle-relabeling transformations, such as (46) or (53). The distinction is important. A genuine two-component fluid does not allow any local relabeling but does allow a global relabeling. Furthermore, conservation of pseudomomentum must be distinguished from the conservation of momentum which follows when the Lagrangian density is invariant under a uniform translation in physical space.

The pseudomomentum vector is not uniquely defined. Different choices of fluid labels lead to different pseudomomenta, and so do different but equivalent (see Appendix D) choices of the Lagrangian density. Because of this nonuniqueness the question becomes what is an optimal choice, with optimal choice usually being understood as one that makes the pseudomomentum quadratic to lowest order in the deviation from a reference state. This reference state can be the resting ocean [e.g., *Ripa*, 1982], a parallel flow [e.g., *Ripa*, 1992], or a time-evolving mean flow [e.g., *Andrews and McIntyre*, 1978a, b]. Our definition (48) is referenced to an inertial frame.

3.3. Canonical Eulerian Formulations

The equations of motion in their standard Eulerian form, as, for example, given in Appendix C, cannot be derived from a Lagrangian density using Hamilton's variational principle. The transformation from the Lagrangian variable $\mathbf{x}(\mathbf{s}, \tau)$ to the Eulerian variable $\mathbf{u}(\mathbf{x}, t)$ is not a canonical transformation. However, there exist other sets of dependent variables that are Eulerian, that is, have $\mathbf{x}$ and t as independent variables and whose equations of motion can be derived from a Lagrangian density using Hamilton's variational principle. Most of the Eulerian forms of Hamilton's principle were originally derived in an ad hoc fashion [e.g., *Lin*, 1963; *Seliger and Whitham*, 1968]. It has only recently been shown by *Salmon* [1988a] that all these Eulerian forms of the variational principle can be derived from the Lagrangian form by considering, instead of the mapping $\mathbf{x} = \mathbf{x}(\mathbf{s}, \tau)$, the inverse mapping $\mathbf{s}$”$\mathbf{u}(\mathbf{x}, t)$ and interchanging dependent and independent variables in the variational principle. Different forms and different representations are obtained depending on which constraints are added to the variational principle. For details, see Appendix E. Here we consider the variational principle that was originally suggested by *Seliger and Whitham* [1968]. It states that

$$\delta \int dt \iiint d^3x\, M\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + e + \Phi + \mathbf{B} \cdot \frac{\partial \mathbf{s}}{\partial t}\right) = 0 \tag{67}$$

for independent variations $\delta\mathbf{B}$ and $\delta\mathbf{s}$. Here $M = \partial(\mathbf{s})/\partial(\mathbf{x})$ is the Jacobian of the inverse mapping and

$$\mathbf{u} + \mathbf{U} = B_1 \nabla s_1 + B_2 \nabla s_2 + B_3 \nabla s_3 \tag{68}$$

The vector $\mathbf{B}$, which enters the variational principle as the Lagrange multiplier associated with the constraint $(D/Dt)\mathbf{s} = 0$ [*Lin*, 1963], is the pseudomomentum vector discussed in the previous section.

For a two-component system, $e = e(v, \eta, c)$. Using the continuity equation in the form

$$v(\mathbf{x}, t) = M^{-1} \tag{69}$$

and the specific entropy and concentration equations in the form

$$\eta(\mathbf{x}, t) = \eta_0(\mathbf{s}) \qquad c(\mathbf{x}, t) = c_0(\mathbf{s}) \tag{70}$$

the specific internal energy becomes $e = e(M^{-1}, \eta_0, c_0)$ where η_0 and c_0 are given functions of $\mathbf{s}$. Variations $\delta\mathbf{B}$ and $\delta\mathbf{s}$ yield the canonical Eulerian equations

$$\frac{D}{Dt}\mathbf{s} = 0 \tag{71a}$$

$$\frac{D}{Dt}\mathbf{B} = T\tilde{\nabla}\eta_0 + \Delta\mu\tilde{\nabla}c_0 - \tilde{\nabla}\Gamma \tag{71b}$$

where

$$\Gamma = \tfrac{1}{2}\mathbf{u} \cdot \mathbf{u} + e + \Phi + pv + \mathbf{B} \cdot \frac{\partial s}{\partial t} \tag{72}$$

is a modified Bernoulli function and where the pressure p, temperature T, and chemical potential difference $\Delta\mu$ are defined by the thermodynamics relations given in Appendix A. Note that the derivatives on the right-hand side of (71b) are with respect to the dependent variable $\mathbf{s}$. The fluid evolution is described by prognostic equations for $\mathbf{B}$ and $\mathbf{s}$ and by diagnostic equations for $\mathbf{u}$, v, η, c, p, T, and $\Delta\mu$. It is an Eulerian description. The independent variables are $\mathbf{x}$ and t. It is a canonical description. The prognostic equations are derived from a Lagrangian density using Hamilton's variational principle. It is straightforward to show that the canonical Eulerian equations (71) are equivalent to the standard Eulerian equations as, for example, given by (C1). Note that the canonical Eulerian description requires diagnostic equations for T and $\Delta\mu$ which are not needed in the standard Eulerian description.

The canonical Eulerian description has the particle label $\mathbf{s}$ and the pseudomomentum $\mathbf{B}$ as dependent variables. These are the variables that appeared in the derivation of Ertel's theorem in the Lagrangian description via Noether's theorem. Conservation of potential vorticity should therefore be an obvious consequence of the equations of motion. Indeed, for a one-component system $\nabla c_0 = 0$. If we choose $s_3 = \eta_0$, the equations for $\mathbf{B}$ reduce to

$$\frac{D}{Dt} B_1 = -\frac{\partial \Gamma}{\partial s_1} \qquad \frac{D}{Dt} B_2 = -\frac{\partial \Gamma}{\partial s_2} \qquad \frac{D}{Dt} B_3 = T - \frac{\partial \Gamma}{\partial s_3} \tag{73}$$

and state the global conservation of the pseudomomentum components B_1 and B_2. As shown in section 3.2, global conservation of B_1 and B_2 immediately implies material conservation of $q = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\eta$. For a homentropic fluid, $\nabla c_0 = 0$ and $\nabla\eta_0 = 0$, and all components of the pseudomomentum vector are globally conserved. This immediately implies the material conservation of the vector $\boldsymbol{\Pi} = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\mathbf{s}$.

The canonical Eulerian description has two noteworthy properties. Material conservation of potential vorticity is an immediate consequence of the equations of motion. The potential vorticity for homentropic (and homogeneous) fluids can explicitly be expressed in terms of system variables.

4. OCEANIC APPROXIMATIONS

We now turn to oceanic motions and present various approximate forms of Ertel's theorem that are used in oceanography. Now $\boldsymbol{\Omega}$ is Earth's rotation rate and Φ the geopotential. Seawater is treated as a two-component system with water and salt being the two components. The concentration of salt is the salinity S. Instead of the specific entropy, oceanographers prefer the use of the potential temperature $\theta(\eta, S) = T(p_*, \eta, S)$, which is the temperature that a fluid particle would attain if moved at constant specific entropy and salinity to a reference pressure p_*. Like η the potential temperature is materially conserved for ideal fluid motion. Technically, oceanographers work in the (p, θ, S) representation, i.e., they use p, θ, and S as independent thermodynamic variables. Oceanographers also prefer the use of the density $\rho = v^{-1}$ over the specific volume. More details on the thermodynamic specifications of seawater can be found in Appendix B.

The reasons for choosing the potential temperature θ over the specific entropy η are twofold. First, θ is more easily related to the quantities that are actually measured, namely, p, T, and S (see (B2)). Second, the specific entropy is only defined up to a linear function of S [*Kamenkovich*, 1977]. As shown in Appendix B, this is of no consequence for changes of the specific entropy, and only these changes have physical significance, but, of course, it induces ambiguities when η is used as a variable.

4.1. Boussinesq Approximation

In studying oceanic motions, one usually employs a set of approximations which is attributed to *Boussinesq* [1903]. This Boussinesq approximation is applied to almost all oceanic motions except sound waves. Here we discuss its various steps following *Müller and Willebrand* [1989]. The Boussinesq approximation starts from the equations of motion in the (p, θ, S) representation as given in Appendix C and consists of four steps. First, a motionless hydrostatically balanced reference state

$$\mathbf{u}_r = 0 \qquad \theta_r = \text{const} \qquad S_r = \text{const} \tag{74}$$

is subtracted out of the equations of motion, whereas the constants are usually taken to be 0°C and 35 practical salinity units (psu), respectively. The reference state contains the large pressure gradients that are caused by the weight of the overlying water column but do not cause any motion. Deviations from the reference state are denoted by a prime, except for $\mathbf{u}$.

The second step is the anelastic approximation that eliminates sound waves by assuming that the flow has velocities and phase speeds much smaller than the speed of sound. The adjustment of the pressure field and velocity divergence therefore happens on timescales much faster than those at the flow. Pressure changes can be neglected, and the pressure equation (C5) reduces to

$$\rho_r \mathbf{u} \cdot \nabla\Phi = \frac{1}{\kappa} \nabla \cdot \mathbf{u} \tag{75}$$

There is no prognostic equation for the pressure any more. The pressure must be determined diagnostically from the constraint (75) on the velocity divergence.

The third step is a geometric approximation which is possible since the scale depth $D = (\rho_r \kappa_r g)^{-1}$ of the oceanic density field is of the order of 200 km and therefore much larger than the ocean depth H_0 or the vertical scale H of the fluid motion. The terms on the left-hand side of (75) are therefore at least H/D times smaller than the individual terms on the right-hand side, leading to the approximation $\nabla \cdot \mathbf{u} = 0$. The condition $H/D \ll 1$ also implies that ρ_r can be replaced by a constant reference density ρ_* in the momentum equation.

The final step of the Boussinesq approximation incorporates the facts that $p' \ll p_r$ and $\rho' \ll \rho_r$ for observed oceanic motions. Explicitly, the Boussinesq equations are given by

$$\rho_*\left(\frac{D}{Dt}\,\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u}\right) = -\nabla p' - \rho'\nabla\Phi \tag{76a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{76b}$$

$$\frac{D}{Dt}\,\theta' = 0 \tag{76c}$$

$$\frac{D}{Dt}\,S' = 0 \tag{76d}$$

$$\rho' = F(p_r,\ \theta_r + \theta',\ S_r + S') - F(p_r,\ \theta_r,\ S_r) \tag{76e}$$

where the equation of state as a function of p, θ, and S is denoted by F. The density depends only on the reference pressure p_r or, equivalently, on the vertical coordinate.

Note that the equation of state still contains compressibility effects. The density ρ' is not materially conserved as can be seen by taking the material derivative of (76e), which yields

$$\frac{D}{Dt}\,\rho' = -[c^{-2}(p_r,\ \theta_r + \theta',\ S_r + S') - c^{-2}(p_r,\ \theta_r\ S_r)]\rho_r\mathbf{u} \cdot \nabla\tilde{\mathbf{\Phi}} \tag{77}$$

where c is the speed of sound.

The name "Boussinesq approximation" is not always used in the same way as it is here. Often, it includes incompressibility. Occasionally, it includes a linearization of the equation of state (76e) with respect to θ' and S' [e.g., *Veronis*, 1973].

4.2. Ertel's Theorem in the Boussinesq Approximation

The derivation of Ertel's theorem in the Boussinesq approximation starts from the momentum equation (76a) and the nondivergence equation (76b). Taking the curl of the momentum equation gives

$$\frac{D}{Dt}\,(\boldsymbol{\omega} + 2\mathbf{\Omega}) = (\boldsymbol{\omega} + 2\mathbf{\Omega}) \cdot \nabla\mathbf{u} + \frac{1}{\rho_*^2}\,(\nabla\rho' \times \nabla p_r) \tag{78}$$

and hence

$$\frac{D}{Dt}\left[\frac{1}{\rho_*}\,(\boldsymbol{\omega} + 2\mathbf{\Omega}) \cdot \nabla\psi\right] = \frac{1}{\rho_*^3}\,J(\rho',\ p_r,\ \psi) \tag{79}$$

for any materially conserved tracer ψ. This is Ertel's theorem in the Boussinesq approximation. The potential vorticity is

$$q = \frac{1}{\rho_*}\,(\boldsymbol{\omega} + 2\mathbf{\Omega}) \cdot \nabla\psi \tag{80}$$

and contains the constant reference density ρ_* (which can be dropped) rather than the actual density ρ. The evolution of the potential vorticity is again given by a Jacobian, in this case $J(\rho',\ p_r,\ \psi)$. Again potential vorticity is materially conserved if the Jacobian vanishes, which is the case if $\rho' = \rho'(p_r,\ \psi)$. In section 2.3 we showed that this is the case if the isopycnals in the $\{\theta,\ S)$ plane do not turn with pressure.

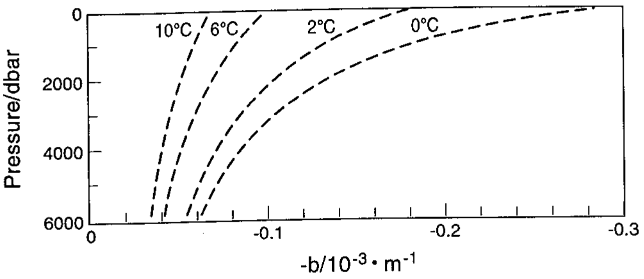

Figure 1. The thermobaric coefficient b of seawater as a function of pressure for various temperatures and a salinity of 35 practical salinity units (psu) [after *Müller and Willebrand*, 1986].

For actual seawater the isopycnals do turn with pressure. The slope of the isopycnals is given by

$$\gamma = \beta/\alpha \tag{81}$$

where α is the thermal expansion coefficient, and β the haline contraction coefficients. The slope changes with pressure according to

$$\left(\frac{\partial\gamma}{\partial p}\right)_{\theta,S} = 2b\gamma \tag{82}$$

where

$$b = \frac{1}{2}\left(\frac{1}{\beta}\,\frac{\partial\beta}{\partial p} - \frac{1}{\alpha}\,\frac{\partial\alpha}{\partial p}\right) \tag{83}$$

is called the thermobaric coefficient [*McDougall*, 1987]. For seawater the thermobaric coefficient is nonzero (Figure 1). The coefficient is called thermobaric since the second term in (83), the change of the thermal expansion coefficient with pressure, dominates for seawater. The coefficient b is negative: The thermal expansion coefficient increases with pressure. The coefficient depends mostly on potential temperature and pressure and only weakly on salinity. It is largest for small temperatures and pressures. The thermobaric coefficient can also be expressed in terms of the adiabatic compressibility κ as

$$b = \frac{1}{2}\left(\frac{1}{\beta}\,\frac{\partial\kappa}{\partial S} + \frac{1}{\alpha}\,\frac{\partial\kappa}{\partial\theta}\right) \tag{84}$$

Again, the second term dominates. The adiabatic compressibility decreases with temperature.

The turning of the isopycnals causes two fluid particles that have different potential temperatures and salinities but the same density on one pressure surface to have different densities on another pressure surface (under ideal fluid motion). There is evidence for this effect in the ocean. The Greenland Sea Water is heavier than the Weddell Sea Water, but when it becomes the North Atlantic Deep Water it overlies the Antarctic Bottom Water originating from the Weddell Sea [*Veronis*, 1973]. Weddell Sea Water is colder than Greenland Sea Water and has a larger compressibility.

Potential vorticity is not materially conserved for seawater, not even for ideal fluid motions. Seawater is a genuine two-component fluid. Oceanographers use, however, a variety of approximations to the equation of state that make seawater a one-component fluid. The first such approximation assumes that the thermobaric coefficient is zero, $b = 0$. In this case the potential density

$$\rho_{\mathrm{pot}}(\theta, S) = F(p_*, \theta, S) \tag{85}$$

is introduced. The potential density is the density a fluid particle would attain if moved at constant θ and S to a reference pressure p_*, usually the surface pressure. The potential density is materially conserved for ideal fluid motion, and it is the materially conserved tracer ψ that makes the potential vorticity q materially conserved for nonthermobaric seawater. The condition $b = 0$ is implemented into the equation of state by changing from the (p, θ, S) representation to the $(p, \rho_{\mathrm{pot}}, S)$ representation (see Appendix B). In the latter representation the condition $b = 0$ is equivalent to

$$\frac{\partial \rho(p, \rho_{\mathrm{pot}}, S)}{\partial S} = 0 \tag{86}$$

The density becomes a function of pressure and potential density only. In the nonthermobaric limit the two prognostic equations for θ' and S' can be replaced by one prognostic equation for ρ_{pot}. The number of prognostic equations is reduced by one. The chemical mode, which is called the temperature-salinity mode in oceanography [e.g., *Müller and Willebrand*, 1986] and which owes its existence to the fact that the dynamical evolution depends on the individual temperature and salinity fields, is eliminated.

The density is still not materially conserved for a nonthermobaric fluid but obeys

$$\frac{D}{Dt}\rho' = -[c^{-2}(p_r, \rho^r_{\mathrm{pot}} + \rho'_{\mathrm{pot}}) - c^{-2}(p_r, \rho^r_{\mathrm{pot}})]\rho_r \mathbf{u} \cdot \nabla\Phi \tag{87}$$

This equation motivates the second approximation that is often made to the equation of state, namely,

$$\frac{\partial c^{-2}(p, \rho_{\mathrm{pot}})}{\partial \rho_{\mathrm{pot}}} = 0 \tag{88}$$

which is equivalent to

$$\frac{\partial \rho'(p, \rho_{\mathrm{pot}})}{\partial p} = 0 \tag{89}$$

and implies $\rho' = \rho'_{\mathrm{pot}}$. The perturbation density is incompressible and hence equal to the perturbation potential density. Now ρ' is materially conserved and can be used as the tracer ψ that makes the potential vorticity materially conserved. This limit is called nearly incompressible, because the reference density is still compressible and the evolution of density is governed by

$$\frac{D}{Dt}\rho = -\rho_r c_r^{-2}\mathbf{u} \cdot \nabla\Phi \tag{90}$$

In the nearly incompressible limit, the perturbation density ρ' can be used as one of the prognostic variables, and no equation of state is needed.

Finally, oceanographers assume seawater to be completely incompressible. Then the density ρ is materially conserved and can be used to form a materially conserved potential vorticity. The three approximations are summarized in Table 2. The densities ρ_{pot}, ρ', or ρ are not materially conserved when heat conduction and salt diffusion are added to the temperature and salinity equations (75c) and (75d).

TABLE 2. Summary of the Materially Conserved Potential Vorticities for Three Approximations to the Equation of State of Seawater

Equation of State	*Form of Materially Conserved Potential Vorticity*
Nonthermobaric ($b = \frac{1}{2}[(1/\beta)(\partial\kappa/\partial S) + (1/\alpha)(\partial\kappa/\partial\theta)] = 0$)	$(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\rho_{\mathrm{pot}}$
Nearly incompressible ($\kappa = \kappa(p)$)	$(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\rho'$
Incompressible ($\kappa = 0$)	$(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\rho$

4.3. Approximate Forms

Here we discuss various approximate forms of Ertel's potential vorticity. These approximate forms make assumptions about the geometry and the scales of the flow. The tracer ψ is ρ_{pot}, ρ', or ρ depending on whether seawater is assumed to be nonthermobaric, nearly incompressible, or incompressible.

Spherical approximation. The fundamental geometric approximation in oceanography is the spherical approximation. It maps the approximate oblate spheroidal shape of the geoid onto a sphere and introduces spherical polar coordinates (φ, θ, and r) where φ is longitude, θ latitude, and r radial distance. Gravity remains exactly perpendicular to r surfaces, i.e., $\nabla\phi$ ×

$\nabla r = 0$ exactly [*Gill*, 1982]. The spherical approximation also assumes that the metric coefficients do not vary with radial distance and that the gravitational acceleration $g = |\nabla\Phi|$ is constant. The spherical approximation represents the lowest order in an expansion of the metric with respect to the two small parameters $d^2/4r_0^2$ and H_0/r_0 where d is the half distance between the foci of the geoid, r_0 the mean radius of Earth, and H_0 the ocean depth. The approximation also introduces the vertical coordinate $z = r - r_0$.

In the spherical approximation the potential vorticity takes the form

$$q = \omega_\varphi \frac{1}{r_0 \cos\theta}\frac{\partial\psi}{\partial\varphi} + \omega_\theta \frac{1}{r_0}\frac{\partial\psi}{\partial\theta} + 2\Omega\cos\theta \frac{1}{r_0}\frac{\partial\psi}{\partial\theta} + \omega_z \frac{\partial\psi}{\partial z} + 2\Omega\sin\theta\frac{\partial\psi}{\partial z} \qquad (91)$$

where

$$\omega_\varphi = \frac{1}{r_0}\frac{\partial w}{\partial\theta} - \frac{\partial v}{\partial z} \qquad (92a)$$

$$\omega_\theta = \frac{\partial u}{\partial z} - \frac{1}{r_0\cos\theta}\frac{\partial w}{\partial\varphi} \qquad (92b)$$

$$\omega_z = \frac{1}{r_0\cos\theta}\frac{\partial v}{\partial\varphi} - \frac{1}{r_0\cos\theta}\frac{\partial(u\cos\theta)}{\partial\theta} \qquad (92c)$$

are the spherical components of the relative vorticity vector and u, v, and w the zonal, meridional, and vertical velocity components, respectively.

Scales. Further approximations depend on the scales of motion. Denote the horizontal length scale by L, the vertical length scale by H, the horizontal velocity scale by U, and the vertical velocity scale by W. These scales define the aspect ratio

$$\delta = H/L, \qquad (93)$$

the (advective) Rossby number

$$\mathrm{Ro} = U/fL \qquad (94)$$

and the geometry parameter

$$\varepsilon = L/r_0 \qquad (95)$$

which measures the extent to which motions feel the spherical geometry of Earth. Here

$$f = f_z = 2\Omega\sin\theta \qquad (96)$$

is the vertical component of the planetary vorticity vector, and is called the Coriolis parameter or frequency. The different terms in the right-hand side of (91) scale as follows: The first term scales as $\mathrm{Ro}\delta^2$ and Ro; the second term as Ro and $\mathrm{Ro}\delta^2$; the third as δ; the fourth as Ro; and the fifth as 1. The Rossby number measures the ratio of the vertical component of the relative vorticity to the vertical component of the planetary vorticity. Nondivergence of the velocity field implies $W = \delta U$.

Shallow water approximation. Motions for which the aspect ratio δ is small are described by the shallow water equations that are given in Appendix F. They are obtained by expanding the equations of motion with respect to δ and keeping only zeroth-order terms. Performing the same expansion in (91), the potential vorticity becomes

$$q = \tilde{\boldsymbol{\omega}}_h \cdot \nabla_h\psi + (\omega_z + f)\frac{\partial\psi}{\partial z} \qquad (97)$$

where

$$\tilde{\boldsymbol{\omega}}_h = \left(-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}\right) \qquad (98)$$

represents the vertical shear and

$$\nabla_h = \left(\frac{1}{r_0\cos\theta}\frac{\partial}{\partial\varphi}, \frac{1}{r_0}\frac{\partial}{\partial\theta}\right) \qquad (99)$$

is the horizontal gradient operator. The meridional component $f_\theta = 2\Omega\cos\theta$ of the planetary vorticity vector does not appear in (97).

Planetary geostrophic motions. Further approximations depend on Ro and ε. Motions with $\mathrm{Ro} \ll 1$ and $\varepsilon = O(1)$ are planetary geostrophic motions. Their potential vorticity is given by

$$q = f\frac{\partial\psi}{\partial z} \qquad (100)$$

These motions will be considered in more detail in section 5.1.

Stratified fluid approximation. Most of the ocean is stably stratified. Often one is interested not in the slow processes that maintain the basic stratification but only in the fast processes that perturb a given stratification. In this case one decomposes the density field

$$\rho(\mathbf{x}, t) = \tilde{\rho}(z) + \delta\rho(\mathbf{x}, t) \qquad (101)$$

where $\tilde{\rho}(z)$ is a prescribed background field and $\delta\rho(\mathbf{x}, t)$ the motionally induced deviation from it. The scale δD of the motion-induced part and the scale $\tilde{D}$ of the background part define the parameter

$$S = \delta D/\tilde{D} \qquad (102)$$

which measures the strength of the stratification. Generally, one assumes $S < 1$ for (101) to be a useful decomposition. For incompressible seawater the density (101) is the appropriate tracer ψ in the potential vorticity expression (97). For nearly incompressible flow

$$\rho'(\mathbf{x}, t) = [\tilde{\rho}(z) - \rho_r(z)] + \delta\rho(\mathbf{x}, t) \qquad (103)$$

is the appropriate tracer. The first term in (103) defines the Brunt-Väisälä or buoyancy frequency squared

$$N^2(z) = -\frac{g}{\rho_*}\frac{d}{dz}[\tilde{\rho}(z) - \rho_r(z)] = -\frac{g}{\rho_*}\frac{d\tilde{\rho}}{dz} - \frac{g^2}{c_r^2} \tag{104}$$

If the motion is furthermore nonconductive, nondiffusive, and nonoverturning, one can introduce the vertical displacement ζ by

$$\rho'(\mathbf{x}, t) = \tilde{\rho}(z - \zeta(\mathbf{x}, t)) - \rho_r(z - \zeta(\mathbf{x}, t)) \tag{105}$$

and have it replace ρ' as a dependent variable. The potential vorticity then becomes

$$q = -\frac{\rho_*}{g} N^2(z - \zeta)\tilde{q} \tag{106}$$

where

$$\tilde{q} = \tilde{\boldsymbol{\omega}} \cdot \nabla_h(z - \zeta) + (\omega_z + f)\frac{\partial}{\partial z}(z - \zeta) \tag{107}$$

The quantity $\tilde{q}$ is the expression for the potential vorticity when ζ is used as a variable. It corresponds to the choice $\psi = z - \zeta$. The potential vorticity $\tilde{q}$ does not contain the high-wavenumber contributions caused by the fine structure of the Brunt-Väisälä frequency. Note that $\tilde{q}$ has a different dimension than q. The parameter S can be interpreted as the scale of the vertical strain $\partial\zeta/\partial z$.

Beta plane approximation. For motions with $\varepsilon \ll 1$ the beta plane approximation is introduced. It is a systematic expansion of the equations of motion around a reference latitude θ_0 with respect to $\varepsilon \sim \theta - \theta_0$. Terms up to first order in ε are kept. The approximation also introduces the horizontal coordinates

$$x = r_0 \cos\theta_0(\varphi - \varphi_0) \qquad y = r_0(\theta - \theta_0) \tag{108}$$

and expands the Coriolis parameter as

$$f = f_0 + \beta_0 y + \cdots \tag{109}$$

where $f_0 = 2\Omega \sin\theta_0$ is the Coriolis parameter and

$$\beta_0 = \frac{1}{r_0}\frac{\partial f}{\partial \theta} = \frac{2\Omega}{r_0}\cos\theta_0 \tag{110}$$

the beta parameter at the reference latitude. The beta parameter describes the changes of the planetary vorticity with latitude. The first term on the right-hand side of (109) scales as 1 and the second as ε.

When implementing the beta plane approximation, the lowest order of the potential vorticity expression becomes $\tilde{q} = f_0$, which is a given constant. The next higher order terms need to be considered, which yield the rather lengthy expression

$$\begin{aligned} q = {} & \omega_z + \frac{y}{r_0}\tan\theta_0 \frac{\partial v}{\partial x} + \frac{u}{r_0}\tan\theta_0 + f_0 + \beta_0 y \\ & - \tilde{\omega}_x \frac{\partial \zeta}{\partial x} - \tilde{\omega}_x \frac{y}{r_0}\tan\theta_0 \frac{\partial \zeta}{\partial x} - \tilde{\omega}_y \frac{\partial \zeta}{\partial y} - \omega_z \frac{\partial \zeta}{\partial z} \\ & - \frac{y}{r_0}\tan\theta_0 \frac{\partial v}{\partial x}\frac{\partial \zeta}{\partial z} - \frac{u}{r_0}\tan\theta_0 \frac{\partial \zeta}{\partial z} - f_0 \frac{\partial \zeta}{\partial z} \\ & - \beta_0 y \frac{\partial \zeta}{\partial z} \end{aligned} \tag{111}$$

where

$$(\tilde{\omega}_x, \tilde{\omega}_y, \omega_z) = \left(-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right) \tag{112}$$

and the 13 terms on the right-hand side of (111) scale as Ro, Roε, Roε, 1, ε, RoS, Ro$S\varepsilon$, RoS, RoS, Ro$S\varepsilon$, Ro$S\varepsilon$, S, and εS, respectively. This expression is correct to first order in ε. Its complexity shows that the beta plane approximation does not simplify matters unless combined with other approximations or applied at the equator, where all the terms proportional to $\tan\theta_0$ vanish. The variables x and y are not Cartesian coordinates but the rescaled spherical coordinates from (108). The equations of motion in the beta plane approximation are given in Appendix F.

The following special cases can be derived from (111).

Quasi-geostrophic motions. The case Ro $\sim \varepsilon \sim S \ll 1$ leads to the quasi-geostrophic limit

$$\tilde{q} = f_0 + \beta_0 y + \omega_z - f_0 \frac{\partial \zeta}{\partial z} \tag{113}$$

where the four terms on the right-hand side scale as 1, ε, Ro, and S, respectively. In this case the potential vorticity consists of the planetary vorticity $f_0 + \beta_0 y$, the relative vorticity ω_z, and the "thermal" vorticity $f_0\, \partial\zeta/\partial z$. This limit will be discussed in more detail in section 5.1.

The *f* plane approximation. When $\varepsilon \ll$ Ro, i.e., $L \ll (U/\beta_0)^{1/2}$ or $\varepsilon \ll S$ then all spherical effects can be neglected. The flow evolves as if occurring on a plane that rotates with constant angular frequency $f_0/2$. The expression for the potential vorticity reduces to

$$q = f_0 + \omega_z - f_0 \frac{\partial \zeta}{\partial z} - \boldsymbol{\omega} \cdot \nabla\zeta \tag{114}$$

where the four terms on the right-hand side scale as 1, Ro, Ro, and RoS, respectively. In this f plane approximation, the shallow water approximation does not need to be applied. The relative vorticity vector in (114) is $\boldsymbol{\omega} = ((\partial w/\partial y) - (\partial v/\partial z), (\partial u/\partial z) - (\partial w/\partial x), (\partial v/\partial x) - (\partial u/\partial y))$, not the shallow water approximation $\tilde{\boldsymbol{\omega}}$ given in (112). The equations of motion in the

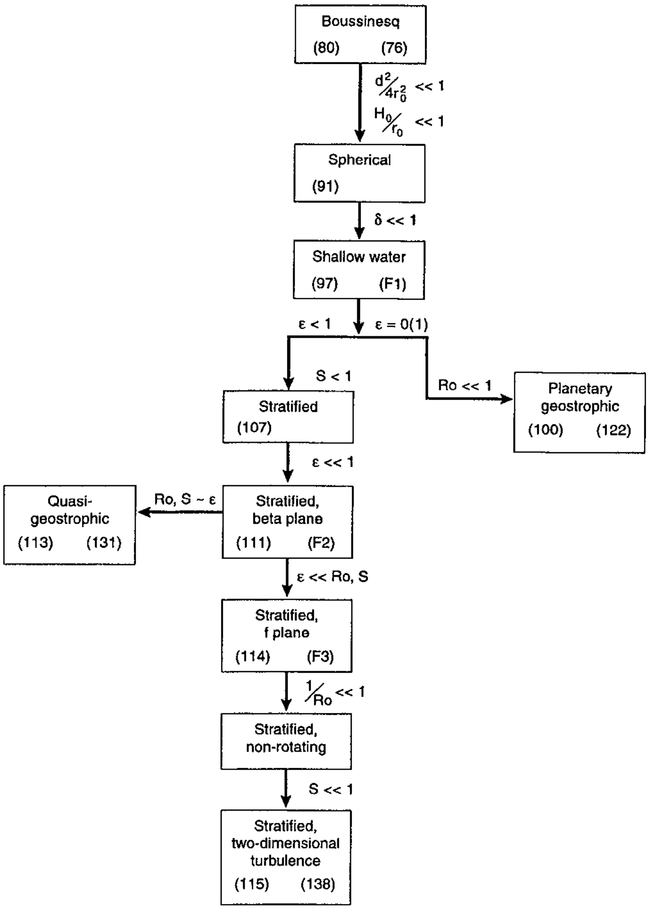

Figure 2. Schematic representation of the most common approximations to the Boussinesq equations. The approximations are characterized by the smallness of the indicated dimensionless parameters. The potential vorticity in a particular approximation is given by the equations whose numbers are in the lower left corner, the equations of motion by the equations whose numbers are in the lower right corner.

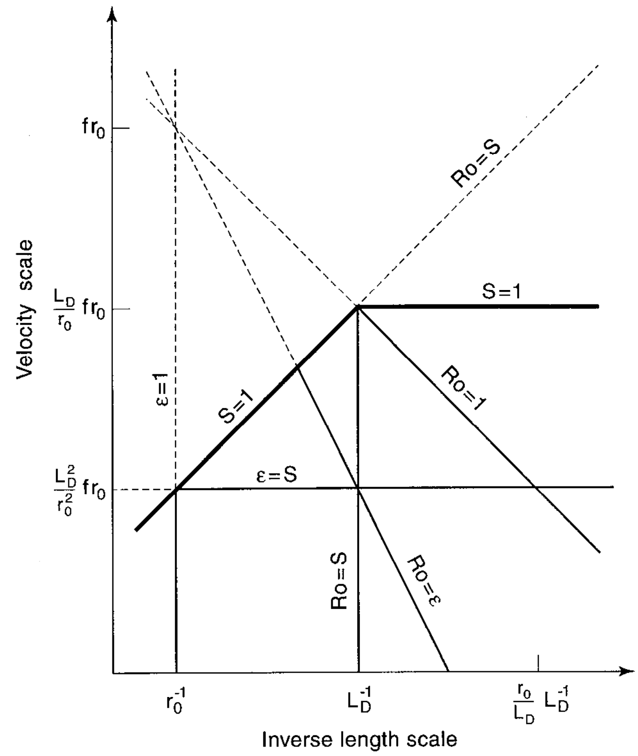

Figure 3. Velocity–inverse length scale diagram for stratified fluids. The diagram shows the characteristic values of the dimensionless parameters Ro, ε, and S that delineate regions where different approximations are valid. The line $S = 1$ separates overturning motions $S \gtrsim 1$ from nonoverturning motions. For nonoverturning motions the lines $\varepsilon = S$ and $\varepsilon = \text{Ro}$ separate the region where the beta plane approximation must be made (to the left) from regions where the f plane approximation is valid. Beyond the line Ro = 1 the flow can be regarded as nonrotating. The line Ro = S separates the region where thermal vorticity is dominant from the region where relative vorticity is dominant.

f plane approximation are listed in Appendix F. For Ro $\gg$ 1, rotation can be neglected altogether. A particular interesting case arises when additionally $S \ll 1$. The potential vorticity then reduces to

$$q = \omega_z \tag{115}$$

and the motion is called stratified two-dimensional turbulence. This case is also discussed more extensively in section 5.1.

For overturning or conductive and diffusive motions, expressions for the potential vorticity must be used that contain the density ρ' rather than the vertical displacement ζ.

The approximations of this section are summarized in Figure 2. Figure 3 additionally shows the characteristic values of the dimensionless parameters Ro, ε, and S that delineate regions where different approximations are valid in $\{U, L^{-1}\}$ space.

An important point is that the various approximate forms of the potential vorticity are indeed materially conserved for the corresponding approximate equations of motion. This can be demonstrated in all cases but needs to be done explicitly. If these approximations were done to the Lagrangian density retaining the particle-relabeling symmetry, then Noether's theorem would automatically insure that the approximate equations conserve the corresponding approximate forms of potential vorticity. Choosing this approach, *Salmon* [1983, 1985, 1988b] arrived at a Lagrangian density and associated equations of motion that are less restrictive than the quasi-geostrophic equations and are a generalization of the semigeostrophic equations introduced by *Hoskins* [1975] for f plane flows in the atmosphere. These generalized semigeostrophic equations are canonical and offer advantages over the quasi-geostrophic equations in that they allow order 1 variations of topography and planetary vorticity. They are algebraically complex since they utilize advected independent variables.

4.4. Isopycnal Coordinates

Here we give expressions for the potential vorticity in isopycnal coordinates. In a stably stratified ocean the density increases monotonically downward with depth and can be used as an independent variable instead of the depth z, which becomes a dependent variable [*Kasahara*, 1974]. The isopycnal coordinate system is thus obtained by the replacements

$$(x, y, z, t) \rightarrow (x, y, \rho, t) \tag{116a}$$

$$\rho = \rho(x, y, z, t) \rightarrow z = z(x, y, \rho, t) \tag{116b}$$

Instead of the density ρ, any other single-valued monotonic function of z can be used as a new independent variable. Choosing the specific entropy leads to the isentropic coordinate system commonly used in meteorology. The introduction of isopycnal coordinates only requires monotony of the density profile. The density does not need to be materially conserved. If it is, then the density can be used as a label coordinate and the isopycnal coordinate system becomes intermediate between the Eulerian and Lagrangian coordinate systems.

In isopycnal coordinates the potential vorticity $q = v(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\rho$ takes the form

$$\begin{aligned} q = v\bigg(& \frac{1}{r_0 \cos\theta}\frac{\partial v}{\partial \varphi} - \frac{1}{r_0 \cos\theta}\frac{\partial (u \cos\theta)}{\partial \theta} \\ & - \frac{1}{r_0}\frac{\partial w}{\partial \theta}\frac{1}{r_0 \cos\theta}\frac{\partial z}{\partial \varphi} + \frac{1}{r_0 \cos\theta}\frac{\partial w}{\partial \varphi}\frac{1}{r_0}\frac{\partial z}{\partial \theta} \\ & + 2\Omega \sin\theta - 2\Omega \cos\theta \frac{1}{r_0}\frac{\partial z}{\partial \theta}\bigg)\left(\frac{\partial z}{\partial \rho}\right)^{-1} \end{aligned} \tag{117}$$

where the partial derivatives with respect to φ and θ are at constant ρ. In the shallow water limit this expression reduces to

$$q = v\,\frac{\omega_\rho + f}{\partial z/\partial \rho} \tag{118}$$

Usually, this expression is rewritten by introducing the height [e.g., *Cushman-Roisin*, 1994]

$$h = \Delta\rho\,\frac{\partial z}{\partial \rho} \tag{119}$$

of the layer between ρ and $\rho + \Delta\rho$ where $\Delta\rho$ is an arbitrary constant density difference. The potential vorticity then takes the form

$$q = v\,\frac{\omega_\rho + f}{h} \tag{120}$$

and represents probably the best known specific expression for the potential vorticity. It was first derived by *Rossby* [1940] in isentropic coordinates.

A layered model consists of a stack of moving layers of uniform density. It results from discretizing the density variable into a finite number of values. Within each homogeneous layer the density cannot be used as a tracer to define a potential vorticity, but the vertical equilibrium position r_3 can. Thus $\partial z/\partial \rho$ needs to be replaced by $\partial z/\partial r_3$ in (118). In the shallow water approximation, $\partial z/\partial r_3$ is independent of depth and given by

$$\frac{\partial z}{\partial r_3} = \frac{h}{h_0} \tag{121}$$

where h is the actual height of the layer and h_0 its equilibrium height. The potential vorticity of each homogeneous layer is therefore given by (120) where h is the actual height of the fluid layer.

5. MODES OF MOTION

In the linear limit the solution of the Boussinesq equations for nonthermobaric seawater can be written as a superposition of vorticity (e.g., Rossby) and inertia-gravity (e.g., Poincaré) waves. These waves are linear manifestations of modes of motion which can also be defined in the nonlinear case: the vortical or potential vorticity carrying mode that becomes a Rossby wave in the linear limit and the zero-potential vorticity or inertia-gravity mode that becomes a Poincaré wave in the linear limit.

5.1. The Vortical Mode

The conservation of potential vorticity is the backbone of the theory of large-scale oceanic motions [e.g., *Pedlosky*, 1987]. This is not because potential vorticity is materially conserved, which it is not if forcing and dissipation are included or if seawater is treated as a genuine two-component fluid, but because potential vorticity (or a variable closely related to it) becomes the sole prognostic variable for these motions. All other fields can be inferred from this variable by diagnostic relations. Physically, the flow evolves on space scales and timescales such that these other variables have adjusted to their equilibrium values (Rossby adjustment). We call this "potential vorticity–carrying" mode of motion the "vortical" mode as advocated by *Müller et al.* [1986]. In oceanography the three most relevant cases of vortical motion are planetary geostrophic motions, quasi-geostrophic motions, and stratified two-dimensional turbulence. In meteorology, additional low–Rossby number models are used, which are intermediate in accuracy between the shallow water equations and the quasi-geostrophic equations. These intermediate models are reviewed by *McWilliams and Gent* [1980]. The deduction of fields such as winds, temperature, and geopotential height from the potential vorticity field is known in meteorology as the invertibility problem [e.g., *Hoskins et al.*, 1985; *McIntyre and Norton*, 1995]. In general, potential vorticity cannot be inverted. It only determines the stream function part of the along-isopycnal velocity, as shown in section 2.1. Additional assumptions about the dynamics must be made. The exact necessary conditions for invertibility are not known. The examples below give sufficient conditions.

Planetary geostrophic motions. Planetary geostrophic motions are characterized by δ, Ro $\ll 1$, and $\varepsilon = O(1)$. Their equations of motion for nearly incompressible seawater take the form

$$\left(\frac{\partial}{\partial t} + \frac{u}{r_0 \cos\theta}\frac{\partial}{\partial \varphi} + \frac{v}{r_0}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right)\rho' = 0 \tag{122a}$$

$$fv = \frac{1}{\rho_*}\frac{1}{r_0 \cos\theta}\frac{\partial p'}{\partial \varphi} \tag{122b}$$

$$fu = -\frac{1}{\rho_*}\frac{1}{r_0}\frac{\partial p'}{\partial \theta} \tag{122c}$$

$$\frac{\partial p'}{\partial z} = -\rho' g \tag{122d}$$

$$f\frac{\partial w}{\partial z} = \beta v \tag{122e}$$

where $\beta = (1/r_0)(\partial f/\partial\theta)$ is the beta parameter. The first equation, the density equation, is the prognostic equation that governs the dynamical evolution of the flow. Forcing and dissipation terms can be added. The other equations are diagnostic relations: the geostrophic approximation to the horizontal momentum balance, the hydrostatic approximation to the vertical momentum balance, and the nondivergence equation, which is written in the form of a vorticity balance.

The density equation is not the only prognostic equation. This can be seen by decomposing the pressure

$$p = \rho_* g\xi + g\int_z^0 dz'\ \rho' \tag{123}$$

where the first part is the "barotropic" component due to the displacement ξ of the surface and the second part the "baroclinic" component due to density fluctuations. The time evolution of the baroclinic part is governed by the density equation (122a); the evolution of the barotropic part is governed by the kinematic surface boundary condition, which in the planetary geostrophic limit takes the form

$$\frac{\partial}{\partial t}\xi = w \qquad z = 0 \tag{124}$$

where again forcing and dissipation terms can be added. Usually, this equation is converted to an equation for the mass transport stream function [e.g., *Hasselmann*, 1982] since lateral boundary conditions are simpler for the stream function than for the surface displacement.

The density defines the potential vorticity $q = f\,\partial\rho'/\partial z$, which is materially conserved for the planetary geostrophic equation (122). The surface displacement enters the equation

$$\left(\partial_t + \frac{u^{bt}}{r_0\cos\theta}\frac{\partial}{\partial\varphi} + \frac{v^{bt}}{r_0}\frac{\partial}{\partial\theta}\right)\hat{q} = -\frac{f}{(H_0 - h)^2}\left(\frac{u^{bc}}{r_0\cos\theta}\frac{\partial}{\partial\varphi} + \frac{v^{bc}}{r_0}\frac{\partial}{\partial\theta}\right)h \tag{125}$$

where

$$\hat{q} = \frac{f}{H_0 - h + \xi} \tag{126}$$

Here $\mathbf{u}^{bt}$ and $\mathbf{u}^{bc}$ are the "barotropic" and "baroclinic" part of the velocity, H_0 the mean depth of the ocean, and h the bottom elevation. The quantity $\hat{q}$ is the potential vorticity for a homogeneous ocean (see (120)). For an ocean of variable density, $\hat{q}$ is not materially conserved. The right-hand side of (126) describes the changes due to the joint effect of baroclinicity and bottom relief, the JEBAR effect [e.g., *Mertz and Wright*, 1992].

Planetary geostrophic motions are vortical motions. The inertia-gravity mode has been filtered out. The planetary geostrophic equations (122), or their generalization to viscous and diffusive flows, are usually used in their steady state form to study the "thermocline problem" [e.g., *Pedlosky*, 1987]. It remains a challenge to define suitable boundary conditions for these equations. The equations are of a highly degenerate hyperbolic type with the vertical axis being a triple characteristic [*Huang*, 1988]. The upstream-downstream concept is fuzzy, and weak solutions exist with discontinuities across characteristics. Solutions that are based on physical reasoning and that elucidate partial aspects are reviewed by *Huang* [1991]. Furthermore, for steady state problems the distinction between prognostic and diagnostic variables becomes blurred, and the potential vorticity loses its special significance. Indeed, ideal thermocline theories [e.g., *Welander*, 1971] are often based on the equations

$$\mathbf{u}\cdot\nabla q = 0 \qquad \mathbf{u}\cdot\nabla\rho' = 0 \qquad \mathbf{u}\cdot\nabla B = 0 \tag{127}$$

where $B = p' + \rho' g z$ is a Bernoulli function. Equations (127) imply

$$q = q(\rho', B) \tag{128}$$

and *Needler*'s [1985] formula

$$\mathbf{u} = \frac{g}{\rho_*} \frac{\mathbf{e}_z \cdot (\nabla\rho' \times \nabla q)}{(\nabla\rho' \times \nabla q) \cdot \nabla\left(f \dfrac{\partial q}{\partial z}\right)} \nabla\rho' \times \nabla q \qquad (129)$$

which determines the velocity field in terms of ρ'. Though this formula shows that density can be inverted, it is of little practical use because it involves second-order derivatives.

Quasi-geostrophic motions. Quasi-geostrophic motions are geostrophically balanced motions in a stratified fluid on a beta plane. They are characterized by $\mathrm{Ro} \sim \varepsilon \sim S \ll 1$. When the shallow water equations are expanded with respect to these parameters, the zeroth-order equations imply the existence of a stream function ψ such that

$$u = -\frac{\partial\psi}{\partial y} \qquad (130a)$$

$$v = \frac{\partial\psi}{\partial x} \qquad (130b)$$

$$w = 0 \qquad (130c)$$

$$\delta p = \rho_* f_0 \psi \qquad (130d)$$

$$\delta\rho = -\frac{\rho_* f_0}{g} \frac{\partial\psi}{\partial z} \qquad (130e)$$

Here $\delta\rho$ and δp are the deviations from the background fields $\bar{\rho}(z)$ and $\bar{p}(z)$, and f_0 is a constant reference value of the Coriolis parameter. The zeroth-order equations thus provide diagnostic equations for the variables u, v, w, δp, and $\delta\rho$ in terms of the stream function ψ. The zeroth order is degenerate in the sense that the nondivergence equation is automatically satisfied and does not yield an equation for ψ.

As shown in textbooks [e.g., *Pedlosky*, 1987], an equation for ψ is obtained from the first order of the perturbation expansion, which implies the quasi-geostrophic potential vorticity equation

$$\frac{D}{Dt} q = 0 \qquad (131)$$

with

$$q = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\psi + f_0 + \beta_0 y \qquad (132)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{\partial}{\partial t} + J(\psi, \ ) \qquad (133)$$

Equation (131) is called the quasi-geostrophic potential vorticity equation because the dynamical evolution of q is governed by small ageostrophic effects: time dependence, advection, vortex stretching, and changes of planetary vorticity. For nonideal motions the effects of forcing, viscosity, and diffusion must be added to that list. The vertical velocity w is zero to lowest order. The first-order ageostrophic correction

$$w^{(1)} = -\frac{f_0}{N^2}\frac{D}{Dt}\frac{\partial\psi}{\partial z} \qquad (134)$$

can be obtained from the first order of the density equation. Note, however, that $w^{(1)}$ does not enter the material derivative (133), which is strictly along the horizontal. The expression (132) for the potential vorticity contains the Coriolis parameter in its beta plane approximation $f_0 + \beta_0 y$. This is the only obvious manifestation of Earth's sphericity. The spherical geometry is also hidden in the horizontal coordinates x and y, which are not Cartesian coordinates but the rescaled spherical coordinates (108). Special cases of the quasi-geostrophic potential vorticity equation can be obtained depending on the relative magnitudes of relative vorticity, planetary vorticity, and thermal vorticity.

Once the potential vorticity is known, the stream function ψ can be determined by inverting the elliptic operator in (132). The needed boundary values can be inferred from the kinematic boundary conditions. At the surface and bottom these conditions take the form

$$\frac{D}{Dt}\left(g\frac{\partial\psi}{\partial z} + N^2\psi\right) = 0 \qquad z = 0 \qquad (135a)$$

$$\frac{D}{Dt}\frac{\partial\psi}{\partial z} = -\frac{N^2}{f_0}\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) \qquad z = -H_0 + h \qquad (135b)$$

and have to be solved simultaneously with (131). They then give the needed boundary values at the surface and the bottom. At horizontal boundaries δA the kinematic boundary condition implies that at each vertical level

$$\frac{\partial\psi}{\partial s} = 0 \qquad \text{on } \delta A \qquad (136)$$

to lowest order and

$$\frac{d}{dt}\oint_{\delta A} ds\,\frac{\partial\psi}{\partial n} = \left(\frac{d}{dt}\oint_{\delta A} d\mathbf{x}\cdot\mathbf{u}\right) = 0 \qquad (137)$$

to first order where the derivatives $\partial/\partial n$ and $\partial/\partial s$ are outwardly normal and parallel to the horizontal boundary. The additional constraint (137) on the circulation has to be imposed since (136) determines ψ only up to an arbitrary function of time [*McWilliams*, 1977; *Pedlosky*, 1987]. The constraint (137) only states that the circulations do not change with time. The circulations themselves need to be prescribed. These prescribed circulations together with (136) give the horizontal boundary values that enable the inversion of the potential vorticity.

Stratified two-dimensional turbulence. Incompressible ideal two-dimensional flow in a horizontal plane is described by a stream function $\psi(x, y, t)$. The evolution of the stream function is governed by the vorticity equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)q = 0 \tag{138}$$

where

$$q = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi \tag{139}$$

Temporal changes of the vorticity are solely governed by horizontal advection. The flow is turbulent and called two-dimensional turbulence. In our context, two-dimensional turbulence represents vortical motion. The vorticity q is a potential vorticity, since $q = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\psi$ with $\psi = z$.

There are two generalizations to geophysical flows. Both rely on the fact that a strong stratification suppresses vertical motion and decouples horizontal layers. The first generalization is obtained from the quasi-geostrophic potential vorticity equation (131) in the limit S, $\varepsilon \ll \text{Ro}$. In this limit the thermal and planetary vorticity become negligible and the quasi-geostrophic potential vorticity equation reduces to (138) with the significant difference that now ψ depends on the vertical coordinate z as well. The flow consists of two-dimensional turbulence evolving independently in each horizontal layer.

The vorticity equation (138) with a z-dependent stream function is also obtained for nonrotating flows in the limit $S \ll 1$ [*Riley et al.*, 1981; *Lilly*, 1983]. Again the flow evolves independently as two-dimensional turbulence in each horizontal layer. In contrast to the quasi-geostrophic case the flow is now in cyclostrophic balance, and the pressure is given by

$$\boldsymbol{\nabla}_h^2 \delta p = -\rho_* \boldsymbol{\nabla}_h \cdot (\mathbf{u}_h \cdot \boldsymbol{\nabla}_h \mathbf{u}_h) \tag{140}$$

Lilly [1983] suggested that this cyclostrophic flow might describe mesoscale motions in the atmosphere. *Müller* [1984] later suggested that it might describe current fine structure in the ocean. *Lilly* [1983] also coined the name "stratified two-dimensional turbulence" for these cyclostrophic motions. We will use this name both for the cyclostrophic and geostrophic case.

5.2. The Zero–Potential Vorticity Mode

Ideal fluids also support motions that do not carry any potential vorticity, that is, motions for which $q = 0$ or constant everywhere and at any time. These zero–potential vorticity motions are identified in this section, both for general fluids and the ocean. The discussion is simplest in canonical Eulerian variables.

Homentropic fluid. Consider a compressible, nonrotating homentropic fluid. In terms of the canonical Eulerian variables **s** and **B** the condition $q = 0$ becomes

$$\tilde{\boldsymbol{\nabla}} \times \mathbf{B} = 0 \tag{141}$$

which implies that **B** can be expressed as the gradient of a potential

$$\mathbf{B} = \tilde{\boldsymbol{\nabla}}\varphi \tag{142}$$

The velocity is then given by

$$u_i = \frac{\partial \varphi}{\partial s_j}\frac{\partial s_j}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} \tag{143}$$

A compressible homentropic flow of zero potential vorticity represents irrotational motion. The canonical Eulerian equation (71b) thus reduces to

$$\frac{\partial \varphi}{\partial t} = -\left(\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + e + \frac{p}{\rho} + \Phi\right) \tag{144}$$

and, together with the continuity equation and a diagnostic relation $e = e(\rho)$, completely describes the flow evolution. The corresponding variational principle was first reported by *Broer* [1974].

One-component fluid. For a one-component fluid, zero potential vorticity is equivalent to

$$(\tilde{\boldsymbol{\nabla}} \times \mathbf{B})_3 = 0 \tag{145}$$

if $s_3 = \eta_0$ is chosen. This implies

$$\mathbf{B}_h = \tilde{\boldsymbol{\nabla}}_h\varphi \tag{146}$$

The "horizontal" components of the vector **B** can be expressed as the horizontal gradient of a scalar φ. The velocity vector is therefore given by

$$\mathbf{u} = \left(B_3 - \frac{\partial \varphi}{\partial s_3}\right) \cdot \boldsymbol{\nabla}s_3 + \boldsymbol{\nabla}\varphi = C_3\boldsymbol{\nabla}s_3 + \boldsymbol{\nabla}\varphi \tag{147}$$

The representation (147) is called a Clebsch representation [*Clebsch*, 1859]. Zero–potential vorticity motions in a one-component fluid are thus motions that allow a (global) Clebsch representation. Their equations of motion are given by

$$\frac{\partial \varphi}{\partial t} + C_3\frac{\partial \eta}{\partial t} = -\left(\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + e + \frac{p}{\rho} + \tilde{\Phi}\right) \tag{148a}$$

$$\frac{D}{Dt}\eta = 0 \tag{148b}$$

$$\frac{D}{Dt}C_3 = T \tag{148c}$$

and have to be augmented by the continuity equation and a diagnostic relation $e = e(v, \eta)$.

Stratified fluid. For an incompressible stratified Boussinesq fluid on an f plane, consider motions for which $q = f_0$, i.e., motions that carry planetary vorticity but no motionally induced potential vorticity. As before, this condition is equivalent to

$$(\tilde{\boldsymbol{\nabla}} \times \mathbf{B})_3 = f_0 \tag{149}$$

or

$$\mathbf{B}_h = \tilde{\boldsymbol{\nabla}}_h \varphi + f_0 r_1 \tilde{\mathbf{e}}_2 \tag{150}$$

where we have chosen the equilibrium position **r** of the fluid particle as a fluid label. The absolute velocity vector is therefore given by

$$\mathbf{u} + \mathbf{U} = f_0 r_1 \boldsymbol{\nabla} r_2 + \left(B_3 - \frac{\partial \varphi}{\partial r_3}\right) \boldsymbol{\nabla} r_3 + \boldsymbol{\nabla} \varphi \tag{151a}$$

or

$$\mathbf{u} + \mathbf{U} = C_2 \boldsymbol{\nabla} r_2 + C_3 \boldsymbol{\nabla} r_3 + \boldsymbol{\nabla} \varphi \tag{151b}$$

where $\mathbf{U} = f_0/2\mathbf{e}_z \times \mathbf{x}$ in the f plane approximation.

Any velocity vector can be represented in the form (151b) [*Boozer*, 1985]. The canonical Eulerian equations for this velocity representation are

$$\frac{D}{Dt} r_i = 0 \qquad i = 2, 3 \tag{152a}$$

$$\frac{D}{Dt} C_2 = 0 \tag{152b}$$

$$\frac{D}{Dt} C_3 = -N^2(r_3)(z - r_3) \tag{152c}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \tag{152d}$$

The nondivergence condition (152d) determines the velocity potential φ. The variational principle for these equations is derived in Appendix C, which also lists the expression for the pressure in terms of the canonical variables r_1, r_2, C_2, C_3, and φ.

For $C_2 = f_0 r_1$, solutions to the above equation inherently have $q = f_0$ and represent nonlinear internal gravity waves which are the manifestations of the inertia-gravity mode in a stratified fluid. In the linear limit

$$q - f_0 \rightarrow \delta q_{\text{lin}} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - f_0 \frac{\partial \eta}{\partial z} \tag{153}$$

and the equations can be recombined to yield the standard linear internal wave equation

$$\left[\left(\frac{\partial^2}{\partial t^2} + f_0^2\right) \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} + (N^2 - f_0^2) \boldsymbol{\nabla}_h \cdot \boldsymbol{\nabla}_h\right] \eta = 0 \tag{154}$$

Flow in the vertical plane. The vertical or inertia-gravity mode of motion can also be suppressed by purely geometrical constraints. In the previous section we showed that two-dimensional flow in a horizontal plane, i.e., two-dimensional turbulence, is purely vortical motion. The counterpart is a two-dimensional incompressible stratified Boussinesq flow in a nonrotating vertical (x, z) plane. The conditions $\partial/\partial y$ and $v = 0$ imply

$$q = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}(z - \zeta) \equiv 0 \tag{155}$$

Two-dimensional motions in a nonrotating vertical plane have zero potential vorticity and represent the inertia-gravity mode of motion.

5.3. Normal Mode Decomposition

Normal mode decompositions are based on the eigenvectors of the linearized dynamical equations. They are an inherently linear concept. As an example, consider motions of an incompressible stratified fluid on an f plane. The equations of motion are given in Appendix F. They contain three prognostic equations. In the linear limit these equations can be cast into the form [e.g., *Lien and Müller*, 1992]

$$\frac{\partial \boldsymbol{\psi}}{\partial t} + \mathcal{H} \boldsymbol{\psi} = 0 \tag{156}$$

with the state vector

$$\boldsymbol{\psi} = \begin{pmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \\[2ex] f_0 \dfrac{\partial \eta}{\partial z} \end{pmatrix} \tag{157}$$

and the system matrix

$$\mathcal{H} = \begin{pmatrix} 0 & -f_0\left(1 - \dfrac{\Delta_h}{\Delta}\right) & \dfrac{-N^2}{f_0} \dfrac{\Delta_h}{\Delta} \\ f_0 & 0 & 0 \\ f_0 & 0 & 0 \end{pmatrix} \tag{158}$$

Here Δ_h is the two-dimensional horizontal and Δ the three-dimensional Laplacian. The vertical velocity and the pressure are given by the diagnostic relations

$$\frac{\partial w}{\partial z} = -\psi_1 \qquad \Delta p = \rho_* f_0 \psi_2 - \rho_* \frac{N^2}{f_0} \psi_3 \tag{159}$$

For an infinite linearly stratified ocean with no top and bottom boundary, (156) has wave solutions of the form

$$\boldsymbol{\psi}(\mathbf{x}, t) = \mathbf{P}_k^s \exp \{i(\mathbf{k} \cdot \mathbf{x} - \omega_k^s t)\} \tag{160}$$

with dispersion relation

$$\omega_{\mathbf{k}}^s = s\sigma \qquad s = 0, +, - \tag{161}$$

where

$$\sigma = +\left(\frac{N^2\alpha^2 + f_0^2\beta^2}{k^2}\right)^{1/2} \tag{162}$$

and polarization vectors

$$\mathbf{P}_\mathbf{k}^0 = \frac{N^2\alpha^2}{k\sigma}\begin{pmatrix} 0 \\ 1 \\ -\dfrac{f_0^2}{N^2}\dfrac{\beta^2}{\alpha^2} \end{pmatrix} \qquad \mathbf{P}_\mathbf{k}^s = \frac{f_0\alpha\beta}{2^{1/2}k\sigma}\begin{pmatrix} -is\dfrac{\sigma}{f_0} \\ 1 \\ 1 \end{pmatrix}$$

$$s = +, - \tag{163}$$

Here $\mathbf{k} = (k_x, k_y, k_z)$ is the wavenumber vector, and α, β, and k the moduli of the horizontal, vertical, and total wavenumber vector, respectively. The frequency $\omega_\mathbf{k}^s$ and the polarization vectors $\mathbf{P}_\mathbf{k}^s$ represent the eigenvalues and eigenvectors of the system matrix $\mathcal{H}$. Since there are three prognostic equations, there are three wave modes, labeled by the index s. The mode $s = 0$ is a steady, horizontally nondivergent and geostrophically balanced flow. It is the linear version of the vortical mode on the f plane. The modes $s = +, -$ are linear internal gravity waves. Their linear motionally induced potential vorticity

$$\delta q_{\text{lin}} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - f_0\frac{\partial \eta}{\partial x} \tag{164}$$

is zero, and they represent the linear version of the inertia-gravity mode of motion.

For each wavenumber vector the three polarization vectors $\mathbf{P}_\mathbf{k}^s$ ($s = 0, +, -$) form a complete basis in $\psi_\mathbf{k}$ space, i.e., in the Fourier space of ψ. Therefore each vector $\psi_\mathbf{k}$ can be decomposed into the three components

$$\psi_\mathbf{k} = a_\mathbf{k}^0\mathbf{P}_\mathbf{k}^0 + a_\mathbf{k}^+\mathbf{P}_\mathbf{k}^+ + a_\mathbf{k}^-\mathbf{P}_\mathbf{k}^- \tag{165}$$

The amplitudes are given by

$$a_\mathbf{k}^s = \tilde{\mathbf{P}}_\mathbf{k}^s \cdot \psi_\mathbf{k} \tag{166}$$

where

$$\tilde{\mathbf{P}}_\mathbf{k}^0 = \frac{N}{\sigma k}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \tilde{\mathbf{P}}_\mathbf{k}^s = \frac{f_0\beta}{2^{1/2}k\alpha\sigma}\begin{pmatrix} i\dfrac{k^2\sigma}{\beta^2 f_0} \\ 1 \\ \dfrac{N^2\alpha^2}{f_0^2\beta^2} \end{pmatrix}$$

$$s = +, - \tag{167}$$

are the adjoint eigenvectors or polarization vectors. The decomposition (165) is a unique decomposition of an arbitrary flow field into a part that carries linear motionally induced potential vorticity δq_{lin} and a part that does not. The normal mode decomposition can trivially be generalized to an ocean of finite depth and nonconstant Brunt-Väisälä frequency by introducing the appropriate vertical eigenfunctions, instead of exp $\{ik_z z\}$. For situations where horizontal and vertical dependence cannot be separated, *Hasselmann* [1970] gives the decomposition in a form that involves polarization and frequency operators.

6. SUMMARY AND CONCLUSIONS

Potential vorticity is a central concept in fluid dynamics and in physical oceanography. We reviewed some of its fundamental theoretical aspects: its conservation, its relation to particle-relabeling symmetries, and its role in defining modes of motion. The major findings and conclusions are these:

Potential vorticity is not a unique concept. First, it has been used to denote a class of quantities. The tracer ψ in the definition of q is any (materially conserved) tracer. Second, potential vorticity has been used to denote a specific member of this class, obtained by specifying the tracer ψ. Usually, the tracer is chosen such that this specific member q becomes materially conserved. Usually, it is clear from the context what potential vorticity refers to.

When considering the conservation of potential vorticity, one has to distinguish between global and material conservation. Potential vorticity is always globally conserved. The local time rate of change of ρq is given by the divergence of a flux vector. The global conservation holds for any tracer, materially conserved or not, and for any fluid, ideal or real. Actually, what is conserved is a substance of which ρq is the density or q the amount per unit mass. Global conservation is a consequence of the fact that the quantity ρq can be written as a divergence.

Material conservation means that the potential vorticity of each fluid particle does not change under fluid motion. It is a much stronger conservation law and holds only under restrictive conditions. The first condition is that the fluid be ideal. Molecular viscosity, conductivity, and diffusivity inevitably destroy the material conservation of potential vorticity. The second condition is a constraint on the equation of the state of the fluid. The fluid must be a one-component fluid. Then there exists a tracer ψ such that the potential vorticity formed with this tracer is materially conserved. There is a slight complication here. Although such a tracer does exist it might not be expressible in terms of the variables used to describe the flow. This happens for homentropic or homogeneous fluid flows described in standard Eulerian variables.

The intrinsic reason for the material conservation of potential vorticity is the physical insignificance of particle labels. The thermodynamic state of a one-component fluid is completely specified by two independent thermodynamic variables. A one-dimensional particle-relabeling transformation can be carried out without changing the thermodynamic state of the fluid. In field theories it is shown that the infinitesimal generators of such invariances or symmetries are conserved quantities. The invariance under particle relabeling causes material conservation of potential vorticity. The proof is simplest in a Lagrangian description of the fluid flow using Noether's theorem. The invariance is a local invariance leading to local or material conservation. If the fluid is homentropic, then its thermodynamic state is completely determined by one thermodynamic variable, and a two-dimensional relabeling transformation can be applied without affecting the thermodynamic state of the fluid. Therefore two independent potential vorticities are materially conserved. The symmetry considerations also identify the physical significance of the tracers ψ that make potential vorticity a materially conserved quantity. They are particle labels. Often physical quantities, such as the specific entropy or density, can be used as such particle labels.

The material conservation of potential vorticity has to be distinguished from the global conservation of pseudomomentum which follows when the thermodynamic specification is invariant under a uniform translation in label space. A two-component fluid conserves globally one component of the pseudomomentum vector but does not materially conserve potential vorticity.

The relation between potential vorticity conservation and particle-relabeling symmetries also explains why potential vorticity is such an enigmatic concept in the standard Eulerian description. Ertel published his theorem nearly 200 years after Euler's "Principes généraux du mouvement des fluides." Particle labels are nonphysical variables. They are eliminated in the standard Eulerian description in favor of physical variables such as the specific volume and entropy, thereby, however, obscuring potential vorticity conservation. The Lagrangian and canonical Eulerian descriptions which retain the particle label as independent or dependent variables, on the other hand, provide a natural and transparent access to potential vorticity conservation but at the price of carrying dynamically inconsequential information.

The second part of this review covered oceanic approximation of Ertel's theorem. Seawater is a nonideal two-component fluid. Potential vorticity is never materially conserved. The thermobaricity of seawater and its molecular viscosity, conductivity, and diffusivity always change the potential vorticity of a fluid particle. The thermobaric effect becomes important for large-scale motions that traverse a significant part of the water column. The molecular effects are most effective at small scales. The thermobaric changes can be suppressed by assuming seawater to be nonthermobaric, nearly incompressible, or incompressible. In these cases the respective choices, $\psi = \rho_{pot}$, ρ', or ρ, will make the potential vorticity materially conserved for ideal fluid motion. The fact that it is the thermobaricity and the molecular dissipation mechanisms that change the potential vorticity remains true even if approximations are applied to the dynamical equations. We considered the Boussinesq, spherical, shallow water, planetary geostrophic, beta plane, stratified fluid, quasi-geostrophic, f plane, and stratified two-dimensional turbulence approximations. What all these approximations do is that they provide specific expressions for the potential vorticity. Material changes of potential vorticity are still caused by thermobaricity and molecular processes.

Potential vorticity owes its central role in physical oceanography to the fact that it defines the vortical mode of motion, which is a dynamical mode that carries potential vorticity and for which potential vorticity or a quantity closely related to it is the sole prognostic variable. Vortical motions include planetary geostrophic flows, quasi-geostrophic flows, and stratified two-dimensional turbulence. They exhibit their own kind of characteristic dynamics, which are described in textbooks of geophysical fluid dynamics.

For vortical motions the potential vorticity can be inverted, that is, all other variables can be inferred from the potential vorticity by diagnostic relationships. The most general set of dynamical equations that allow such an inversion are not known. Even for planetary and quasi-geostrophic flows the inversion is complicated. Boundary conditions might not be given or might contain additional prognostic variables.

Potential vorticity is also important in oceanography because it defines nonlinear internal gravity waves in a stratified fluid, that is, the inertia-gravity mode of motion. Nonlinear internal gravity waves are motions that do not possess any motionally induced potential vorticity. This condition immediately leads to a reduction of the number of prognostic variables in the canonical Eulerian descriptions but not so in the standard Eulerian description. In the linear limit this distinction between canonical and standard Eulerian description disappears.

A linearized form of potential vorticity enters the normal mode decomposition of an arbitrary flow into a part that carries linear (motionally induced) potential vorticity and a part that does not. This decomposition can be performed for any flow. It is most useful for linear or nearly linear flows. For nonlinear flows it is not a decomposition into the vortical and inertia-gravity mode. Since decomposition is an inherently linear concept, it is unlikely that a general gravity-vortical mode decomposition exists.

This review has covered fundamental theoretical aspects of potential vorticity and its conservation. It has not covered any applications or phenomenological aspects. Many of the fundamental applications of potential vorticity conservation are described in textbooks. Phenomenological aspects such as the distribution of potential vorticity in the world's ocean, its variability, and spectral distribution represent a major challenge for the future. Among the theoretical aspects the invertibility and the implications of global conservation are major open questions.

APPENDIX A: THERMODYNAMIC SPECIFICATIONS OF N-COMPONENT SYSTEMS

Gibbs's rule states that the thermodynamic state of an N-component system, that is, one consisting of N different components, is completely specified by the values of $N + 1$ thermodynamic variables. Different choices can be made for these independent variables and lead to different thermodynamic representations. This appendix describes the two thermodynamic representations used in this paper.

The $\{v, \eta, c_2, \cdots, c_N\}$ Representation

In this representation the specific volume v, the specific entropy η, and the concentrations $c_2, \cdots, c_N$ are chosen as the $N + 1$ independent thermodynamic variables. Only $N - 1$ concentrations can be used as independent variables since the concentrations sum to 1. For this choice of variables the specific internal energy

$$e = e\{v, \eta, c_2, \cdots, c_N\} \tag{A1}$$

is the thermodynamic potential and completely describes the system. Specifically, the pressure p, the temperature T, and the chemical potential differences $\Delta\mu_i = \mu_i - \mu_1$ $(i = 2, \cdots, N)$ are given by the first derivatives

$$p = -\left(\frac{\partial e}{\partial v}\right)_{\eta, c_2, \cdots, c_N} \tag{A2a}$$

$$T = \left(\frac{\partial e}{\partial \eta}\right)_{v, c_2, \cdots, c_N} \tag{A2b}$$

$$\Delta\mu_i = \left(\frac{\partial e}{\partial c_i}\right)_{v, \eta, c_2, \cdots, c_N} \qquad i = 2, \cdots, N \tag{A2c}$$

where the subscripts denote the variables that are being held constant in the partial differentiation. The individual chemical potentials μ_i $(i = 1, \cdots, N)$ can be inferred from (A2c) and Euler's identity

$$e + pv - T\eta = \sum_{i=1}^{N} \mu_i c_i \tag{A3}$$

The $\{v, \eta, c_2, \cdots, c_N\}$ representation cannot be used for incompressible fluids since (A2a) does not give a unique value for p.

The $\{p, \eta, c_2, \cdots, c_N\}$ Representation

In this representation the pressure p, the specific entropy η, and the concentrations $c_2, \cdots, c_N$ are used as independent variables. The specific enthalpy

$$h = e + pv \tag{A4}$$

is the thermodynamic potential with

$$v = \left(\frac{\partial h}{\partial p}\right)_{\eta, c_2, \cdots, c_N} \tag{A5a}$$

$$T = \left(\frac{\partial h}{\partial \eta}\right)_{v, c_2, \cdots, c_N} \tag{A5b}$$

$$\Delta\mu_i = \left(\frac{\partial h}{\partial c_i}\right)_{p, \eta, c_2, \cdots, c_N} \qquad i = 2, \cdots, N \tag{A5c}$$

The relation $v = v(p, \eta, c_2, \cdots, c_N)$ is the equation of state. The adiabatic compressibility κ, the adiabatic temperature gradient Γ, and the adiabatic chemical potential gradients D_i are defined by the derivatives

$$\kappa = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_{\eta, c_2, \cdots, c_N} \tag{A6a}$$

$$\Gamma = \left(\frac{\partial v}{\partial \eta}\right)_{p, c_2, \cdots, c_N} = \left(\frac{\partial T}{\partial p}\right)_{\eta, c_2, \cdots, c_N} \tag{A6b}$$

$$D_i = \left(\frac{\partial v}{\partial c_i}\right)_{p, \eta c_2, \cdots, c_N} = \left(\frac{\partial \Delta\mu_i}{\partial p}\right)_{\eta, c_2, \cdots, c_N} \qquad i = 2, \cdots, N \tag{A6c}$$

The second equality in (A6b) and (A6c) arises from the fact that the order of differentiation can be changed.

APPENDIX B: THERMODYNAMIC SPECIFICATIONS OF SEAWATER

Seawater can be treated as a two-component fluid consisting of water and salt. The concentration of salt is the salinity S. Various thermodynamic representations are used in oceanography. This appendix describes the two representations used in this paper and an indeterminacy of the specific entropy.

The $\{p, \theta, S\}$ Representation

When a fluid particle is moved at constant specific entropy and salinity across a pressure surface, its temperature changes at a rate given by the adiabatic temperature gradient Γ. To remove this pressure effect on temperature, oceanographers introduce the potential temperature

$$\theta(\eta, S) = T(p_*, \eta, S) \tag{B1}$$

which is the temperature a fluid particle would attain if moved at constant η and S to a reference pressure p_*. Since p, T, and S are usually measured, the equivalent expression

$$\theta(p, T, S) = T + \int_p^{p_*} dp'\, \Gamma[p', \eta(p, T, S), S] \tag{B2}$$

is used for the actual calculation of θ. Processes for which η and S remain constant do not change the potential temperature.

The equation of state in the $\{p, \theta, S\}$ representation is given by

$$\rho = \rho(p, \theta, S) \tag{B3}$$

The derivatives

$$\kappa = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_{\theta,S} \qquad \alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \theta}\right)_{p,S} \qquad \beta = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial S}\right)_{p,\theta} \tag{B4}$$

define the adiabatic compressibility κ, the thermal expansion coefficient α, and haline contraction coefficient β. The equation of state is nonlinear in the sense that the coefficients κ, α, and β depend on p, θ, and S. Among the six independent second-order derivatives, two are of particular importance. The first one is the thermobaric coefficient

$$b = \frac{1}{2}\left(\frac{1}{\beta}\frac{\partial \beta}{\partial p} - \frac{1}{\alpha}\frac{\partial \alpha}{\partial p}\right) = \frac{1}{2}\left(\frac{1}{\beta}\frac{\partial \kappa}{\partial S} + \frac{1}{\alpha}\frac{\partial \kappa}{\partial \theta}\right) \tag{B5}$$

which describes changes of the isopycnal slope $\gamma = \beta/\alpha$ with pressure

$$\left(\frac{\partial \gamma}{\partial p}\right)_{\theta,S} = 2b\gamma \tag{B6}$$

The second one is the cabbeling coefficient

$$d = \frac{1}{\beta}\frac{\partial \beta}{\partial \theta} - \frac{1}{\alpha}\frac{\partial \alpha}{\partial \theta} + \frac{\alpha}{\beta^2}\frac{\partial \beta}{\partial S} - \frac{1}{\beta}\frac{\partial \alpha}{\partial S} \tag{B7}$$

which describes changes of the isopycnal slope along isopycnals

$$\left(\frac{\partial \gamma}{\partial \theta}\right)_{\rho,p} = d\gamma \tag{B8}$$

The $\{p, \rho_{\mathrm{pot}}, S\}$ Representation

In this representation the potential temperature is replaced by the potential density

$$\rho_{\mathrm{pot}}(p, \theta, S) = \rho(p_*, \theta, S) \tag{B9}$$

which is the density a fluid particle would attain if moved at constant θ and S to a reference pressure p_*. Processes for which θ and S are constant do not change the potential density. The equation of state is given by

$$\rho = \rho(p, \rho_{\mathrm{pot}}, S) \tag{B10}$$

and its derivatives are

$$\left(\frac{\partial \rho}{\partial p}\right)_{\rho_{\mathrm{pot}},S} = \rho\kappa \qquad \left(\frac{\partial \rho}{\partial \rho_{\mathrm{pot}}}\right)_{p,S} = \frac{\rho}{\rho_{\mathrm{pot}}}\frac{\alpha}{\alpha(p_*)}$$
$$\left(\frac{\partial \rho}{\partial S}\right)_{p,\rho_{\mathrm{pot}}} = \rho\alpha\left(\frac{\beta}{\alpha} - \frac{\beta(p_*)}{\alpha(p_*)}\right) \tag{B11}$$

If the thermobaric coefficient b is zero, the slope β/α does not depend on p and $(\partial\rho/\partial S)_{p,\rho_{\mathrm{pot}}} = 0$.

Specific Entropy

For seawater there exist empirical formulae for (1) $\rho = \rho(p, T, S)$ (equation of state), (2) $c_p = c_p(p_*, T, S)$ (specific heat at constant pressure), and (3) $\partial\mu/\partial S = (\partial\mu/\partial S)(p, T_{b,f}, S)$, where T_b is the boiling and T_f the freezing temperature and p_* a standard pressure [*Fofonoff*, 1962; *Kamenkovich*, 1977]. From these empirical formulae the specific entropy can be determined up to a linear function of salinity [*Kamenkovich*, 1977]. This indeterminacy does not affect changes of the specific entropy. Using $\eta = \eta(e, \rho, S)$ these changes become

$$\frac{d}{dt}\eta = \left(\frac{\partial \eta}{\partial e}\right)_{\rho,S}\frac{d}{dt}e + \left(\frac{\partial \eta}{\partial \rho}\right)_{e,S}\frac{d}{dt}\rho + \left(\frac{\partial \eta}{\partial S}\right)_{e,\rho}\frac{d}{dt}S \tag{B12}$$

and are invariant when η is replaced by $\eta + a + bS$.

APPENDIX C: IDEAL FLUID EQUATIONS IN STANDARD EULERIAN FORM

This appendix lists the ideal fluid equations of an N-component fluid in their standard Eulerian form. The adjective "Eulerian" implies that the independent variables are the position $\mathbf{x}$ and time t. The adjective "standard" implies that the dependent variables are the velocity $\mathbf{u}$ and any complete set of $N + 1$ thermodynamic variables. According to the choice of thermodynamic variables, different sets of equations are obtained.

The $\{\mathbf{u}, v, \eta, c_2, \cdots, c_N\}$ Representation

In this representation the ideal fluid flow is described by prognostic equations for the velocity $\mathbf{u}$, the specific volume v, the specific entropy η, and the concentrations $c_2, \cdots, c_N$. These prognostic equations are

$$\frac{D}{Dt}\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{\Omega} \times \mathbf{U} = -v\nabla p - \nabla\Phi_g \qquad \text{(C1a)}$$

$$\frac{D}{Dt}v = v\nabla \cdot \mathbf{u} \qquad \text{(C1b)}$$

$$\frac{D}{Dt}\eta = 0 \qquad \text{(C1c)}$$

$$\frac{D}{Dt}c_i = 0 \qquad i = 2, \cdots, N \qquad \text{(C1d)}$$

The momentum balance (C1a) is written in a frame rotating with constant angular velocity $\mathbf{\Omega}$. The velocity $\mathbf{u}$ is the velocity relative to the rotating frame; the velocity $\mathbf{U} = \mathbf{\Omega} \times \mathbf{x}$ is the velocity of the frame at position $\mathbf{x}$. The acceleration terms on the left-hand side include the Coriolis and centripetal accelerations. The centripetal acceleration can be expressed as a gradient of a potential

$$\mathbf{\Omega} \times \mathbf{U} = \nabla\Phi_c \qquad \text{(C2)}$$

with

$$\Phi_c = -\tfrac{1}{2}\mathbf{U} \cdot \mathbf{U} \qquad \text{(C3)}$$

Usually, Φ_c is combined with the external (gravitational) potential Φ_g to form the total potential $\Phi = \Phi_c + \Phi_g$ and (C1a) reduces to (2). The pressure p that appears in the momentum balance is determined by a thermodynamic relation

$$p = p(v, \eta, c_2, \cdots, c_N) \qquad \text{(C4)}$$

which is a diagnostic relation. Note that neither the temperature T nor the chemical potential differences $\Delta\mu_i$ are needed in this representation but can, of course, be determined if the specific internal energy $e = e(v, \eta, c_2, \cdots, c_N)$ is known. The $\{\mathbf{u}, v, \eta, c_2, \cdots, c_N\}$ representation cannot be used for incompressible ($\kappa = 0$) fluids since no diagnostic relation exists for p.

The $\{\mathbf{u}, p, \eta, c_2, \cdots, c_N\}$ Representation

In this representation, $\mathbf{u}$, p, η, and $c_2, \cdots, c_N$ are chosen as the prognostic variables. The continuity equation (C1b) is replaced by

$$\frac{D}{Dt}p = -\frac{1}{\kappa}\nabla \cdot \mathbf{u} \qquad \text{(C5)}$$

which is a prognostic equation for p, and the specific volume is determined by the equation of state

$$v = v(p, \eta, c_2, \cdots, c_N), \qquad \text{(C6)}$$

which is a diagnostic equation. The momentum balance (C1a) and the equations (C1c) and (C1d) for the specific entropy and concentrations remain unchanged.

Incompressible Fluids

Incompressible fluids have zero adiabatic compressibility κ. This limit is most easily performed in the $\{\mathbf{u}, p, \eta, c_2, \cdots, c_N)\}$ representation. The pressure equation (C5) reduces to the nondivergence condition

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{(C7)}$$

and (C1c), (C1d), and (C6) can be combined into

$$\frac{D}{Dt}v = 0 \qquad \text{(C8)}$$

The specific volume is materially conserved. Equations (C7) and (C8) together with the momentum balance (C1a) form a complete set of equations of motion.

APPENDIX D: LAGRANGIAN DESCRIPTION OF IDEAL FLUID MOTION

Kinematics

In the Lagrangian description the fluid motion is described by the position $\mathbf{x}$ of a fluid particle as a function of its label $\mathbf{s}$ and time τ

$$\mathbf{x} = \mathbf{x}(\mathbf{s}, \tau) \qquad \text{(D1)}$$

Often the initial or equilibrium position $\mathbf{r}$ is used as a label. The fluid velocity is given by

$$\mathbf{u}(\mathbf{x}, \tau) = \frac{\partial}{\partial\tau}\mathbf{x}(\mathbf{s}, \tau) \qquad \text{(D2)}$$

To avoid ambiguities, time is denoted by τ in the Lagrangian frame and by t in the Eulerian frame. Similarly, ∇ denotes the gradient in $\mathbf{x}$ space and $\tilde{\nabla}$ the gradient in $\mathbf{s}$ space. Lagrangian and Eulerian time derivatives are related by

$$\frac{\partial}{\partial\tau}f(\mathbf{s}, \tau) = \frac{D}{Dt}f(\mathbf{x}, t) \qquad \text{(D3)}$$

The Lagrangian specification (D1) represents a time-dependent mapping from label space to position space. The Jacobian of this mapping is

$$K \doteq \frac{\partial(\mathbf{x})}{\partial(\mathbf{s})} \qquad \text{(D4)}$$

Eulerian and Lagrangian descriptions are equivalent. To transform from the Lagrangian description $\mathbf{x}(\mathbf{s}, \tau)$ to the Eulerian description $\mathbf{u}(\mathbf{x}, t)$, one has to differentiate $\mathbf{x}(\mathbf{s}, \tau)$ with respect to τ to obtain $\mathbf{u}(\mathbf{s}, \tau)$, invert $\mathbf{x}(\mathbf{s}, \tau)$ to obtain $\mathbf{s} = \mathbf{s}(\mathbf{x}, t)$, and substitute $\mathbf{s} = \mathbf{s}(\mathbf{x}, t)$ into $\mathbf{u}(\mathbf{s}, \tau)$ to obtain $\mathbf{u}(\mathbf{x}, t)$. To transform from the Eulerian to the Lagrangian description, one has to solve the set of ordinary differential equations

$$\frac{d}{dt}\,\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t), t) \tag{D5}$$

subject to the initial conditions $\mathbf{x}(t = 0) = \mathbf{r}$.

Equations of Motion

In the Lagrangian description the equations of motion take the form

$$\ddot{\mathbf{x}} + 2\mathbf{\Omega} \times \dot{\mathbf{x}} = -v\nabla p - \nabla\Phi \tag{D6a}$$

$$v(\mathbf{s}, \tau) = K \tag{D6b}$$

$$\eta(\mathbf{s}, \tau) = \eta_0(\mathbf{s}) \tag{D6c}$$

$$c_i(\mathbf{s}, \tau) = c_{i0}(\mathbf{s}) \qquad i = 2, \cdots, N \tag{D6d}$$

$$p = p(v, \eta, c_2, \cdots, c_N) \tag{D6e}$$

where the overdot denotes differentiation with respect to time τ and the subscript zero prescribed initial fields. In (D6b) the label $\mathbf{s}$ is chosen such that the initial density is uniform in label space, that is, $ds_1 ds_2 ds_3 = \rho_0(\mathbf{r}) dr_1 dr_2 dr_3$. In the Lagrangian frame the momentum equation becomes a differential equation that is second order in time, whereas the continuity, specific entropy, and concentration equations become diagnostic equations. The pressure equation remains a diagnostic equation.

Hamilton's Principle

The momentum equation (D6a) can be derived from a Lagrangian density using Hamilton's variational principle. In general, the Lagrangian density

$$\mathcal{L} = \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, \mathcal{D}, \mathbf{s}, \tau) \tag{D7}$$

is a function of position $\mathbf{x}$, velocity $\dot{\mathbf{x}}$, displacement gradient tensor $\mathcal{D}$ with components $D_{ij} = \partial x_j/\partial s_i$, and the independent variables $\mathbf{s}$ and τ. Hamilton's principle states that the action is an extremum for actual fluid motions, that is,

$$\delta \int d\tau \iiint d^3s\ \mathcal{L} = 0 \tag{D8}$$

for variations $\delta\mathbf{x}$ that vanish on the boundaries. This principle implies the Euler equations

$$\frac{\partial}{\partial\tau}\frac{\delta\mathcal{L}}{\delta x_{i,\tau}} + \frac{\partial}{\partial s_j}\frac{\delta\mathcal{L}}{\delta s_{i,j}} - \frac{\delta\mathcal{L}}{\delta x_i} = 0 \tag{D9}$$

where we have used index notation for the derivatives and the symbol δ to denote the functional derivative. The Euler equations constitute the differential equations of motion for the system.

The specific momentum equation (D6a) can be derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \dot{\mathbf{x}}\cdot\mathbf{U} - e(v, \eta, c_2, \cdots, c_N) - \Phi \tag{D10}$$

where e is the specific internal energy. The Lagrangian density is the kinetic energy density minus the internal and potential energy density. For a two-component system $e = e(v, \eta, c)$, for a one-component system $e = e(v, \eta)$, and for a homentropic fluid $e = e(v)$, respectively. In all cases the pressure is given by $p = -(\partial e/\partial v)$. The Lagrangian density (D10) is not unique. A total divergence $\Delta\mathcal{L} = \partial/\partial\tau\ \mathcal{A}(\mathbf{x}, \mathbf{s}, \tau) + \partial/\partial s_j\ \mathcal{B}_j(\mathbf{x}, \mathbf{s}, \tau)$ can be added without changing the equations of motion.

APPENDIX E:
EULERIAN FORMS OF HAMILTON'S PRINCIPLE

The Eulerian forms of Hamilton's principle use position $\mathbf{x}$ and time t as independent variables. Most forms were originally derived in an ad hoc fashion. It has only recently been shown by *Salmon* [1988a] that all Eulerian forms can systematically be derived from the Lagrangian form. Here we follow Salmon's approach.

Eulerian Variational Principle

The Lagrangian form of Hamilton's principle states

$$\delta \int d\tau \iiint d^3s\ \left(\tfrac{1}{2}\,\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \dot{\mathbf{x}}\cdot\mathbf{U} - e - \Phi\right) = 0 \tag{E1}$$

for arbitrary variations $\delta\mathbf{x}(\mathbf{s}, \tau)$ that vanish on the boundary. Instead of the mapping $\mathbf{x} = \mathbf{x}(\mathbf{s}, \tau)$ from label into physical space, consider the inverse mapping $\mathbf{s} = \mathbf{s}(\mathbf{x}, t)$ from physical into label space. Interchanging dependent and independent variables in (E1) results in

$$\delta \int dt \int d^3x\ M\left(\tfrac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{U} - e - \Phi\right) = 0 \tag{E2}$$

for arbitrary variations $\delta\mathbf{s}(\mathbf{x}, t)$. Here

$$M = \frac{\partial(\mathbf{s})}{\partial(\mathbf{x})} = K^{-1} \tag{E3}$$

is the Jacobian of the inverse mapping. The velocity $\mathbf{u}$ has to be determined from the *Lin* [1963] constraint

$$\frac{D}{Dt}\,\mathbf{s} = 0 \tag{E4}$$

and is given by

$$u_i = -G_{ji}^{-1}\,\frac{\partial s_j}{\partial t} \qquad i = 1, 2, 3 \tag{E5}$$

where $G_{ij} = \partial s_j/\partial x_i$ are the components of the displacement gradient tensor $\mathscr{G}$ of the inverse mapping.

The variational principle (E2) with (E3) and (E5) is the Eulerian form of Hamilton's principle. For a two-component system it has to be augmented by the continuity, entropy, and concentration equations which take the form

$$v(\mathbf{x}, t) = M^{-1} \qquad \eta(\mathbf{x}, t) = \eta_0(\mathbf{s}) \qquad c(\mathbf{x}, t) = c_0(\mathbf{s}) \tag{E6}$$

where the subscript zero denotes given initial conditions. The specific internal energy in the variational principle (E2) is therefore of the form $e = e(M^{-1}, \eta_0, c_0)$. More customary forms of Hamilton's principle are obtained by adding constraints and using canonical conjugate variables.

The B Representation

Here the constraint (E4) is added to the variational principle (E2) with associated Lagrange multiplier $\mathbf{B}$. One obtains the intermediate form

$$\delta \int dt \iiint d^3x \cdot M\left(\tfrac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{U} - e - \Phi - \mathbf{B}\cdot\frac{D\mathbf{s}}{Dt}\right) = 0 \tag{E7}$$

where $\mathbf{s}$, $\mathbf{u}$, and $\mathbf{B}$ are to be varied independently. Variations $\delta\mathbf{u}$ yield the velocity representation

$$\mathbf{u} + \mathbf{U} = B_1\nabla s_1 + B_2\nabla s_2 + B_3\nabla s_3 \tag{E8}$$

The Lagrange multiplier $\mathbf{B}$ is therefore the pseudomomentum discussed in section 3.1. Substitution of (E8) into (E7) yields *Seliger and Whitham*'s [1968] principle

$$\delta \int dt \iiint d^3x\, M\left\{\tfrac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + e + \Phi + \mathbf{B}\cdot\frac{\partial \mathbf{s}}{\partial t}\right\} = 0 \tag{E9}$$

where $\mathbf{s}$ and $\mathbf{B}$ are canonical conjugate pairs and are to be varied independently. The vector $\mathbf{u}$ is just an abbreviation for (E8). Variations $\delta\mathbf{B}$ and $\delta\mathbf{s}$ yield the equations of motion (71).

The C Representation

A more common form is obtained when the continuity equation is added to the variational principle (E7) with an associated Lagrange multiplier φ. Then

$$\delta \int dt \iiint d^3x \left[\rho\left(\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{U} - e - \Phi - \mathbf{C}\cdot\frac{D\mathbf{s}}{Dt}\right) + \varphi\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right)\right] = 0 \tag{E10}$$

where $\mathbf{s}$, $\mathbf{u}$, $\mathbf{C}$, ρ, and φ are to be varied independently. Variations $\delta\mathbf{u}$ yield the velocity representation

$$\mathbf{u} + \mathbf{U} = C_2\nabla s_2 + C_3\nabla s_3 + \nabla\varphi \tag{E11}$$

where C_1 has been set to zero for reasons discussed by *Salmon* [1988a] and *Boozer* [1985]. The vector $\mathbf{C}$ is a pseudomomentum but differs from $\mathbf{B}$. The Lagrange multiplier φ is a velocity potential. When the velocity representation (E11) is substituted into the variational principle (E10), it takes the form

$$\delta \int dt \iiint d^3x\, \rho\left(\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + e + \Phi + C_2\frac{\partial s_2}{\partial t} + C_3\frac{\partial s_3}{\partial t} + \frac{\partial\varphi}{\partial t}\right) = 0 \tag{E12}$$

where (s_2, C_2), (s_3, C_3) and (ρ, φ) represent canonical conjugate pairs and are to be varied independently. The vector $\mathbf{u}$ is an abbreviation for (E11). A variational principle for incompressible fluids can be obtained by setting $\rho = \rho_0(\mathbf{s})$ in (E12).

APPENDIX F: EQUATIONS OF OCEANIC MOTIONS

Shallow Water Equations

The shallow water equations are obtained by expressing the Boussinesq equations of motion (76) in spherical coordinates and expanding them with respect to δ, keeping only zeroth-order terms. They are

$$\frac{\partial u}{\partial t} + \frac{u}{r_0\cos\theta}\frac{\partial u}{\partial\varphi} + \frac{v}{r_0}\frac{\partial v}{\partial\theta} + w\frac{\partial u}{\partial z} - \frac{uv}{r_0}\tan\theta - fv = -\frac{1}{\rho_* r_0\cos\theta}\frac{\partial p}{\partial\varphi} \tag{F1a}$$

$$\frac{\partial v}{\partial t} + \frac{u}{r_0\cos\theta}\frac{\partial v}{\partial\varphi} + \frac{v}{r_0}\frac{\partial v}{\partial\theta} + w\frac{\partial v}{\partial z} + \frac{u^2}{r_0}\tan\theta + fu = -\frac{1}{\rho_*}\frac{\partial p}{\partial\theta} \tag{F1b}$$

$$0 = \frac{\partial p'}{\partial z} + \rho' g \tag{F1c}$$

$$\frac{1}{r_0\cos\theta}\frac{\partial u}{\partial\varphi} + \frac{1}{r_0\cos\theta}\frac{\partial(v\cos\theta)}{\partial\theta} + \frac{\partial w}{\partial z} = 0 \tag{F1d}$$

$$\frac{\partial\rho'}{\partial t} + \frac{u}{r_0\cos\theta}\frac{\partial\rho'}{\partial\varphi} + \frac{v}{r_0}\frac{\partial\rho'}{\partial\theta} + w\frac{\partial\rho'}{\partial z} = 0 \tag{F1e}$$

The vertical momentum balance reduces to the hydrostatic balance (hydrostatic approximation). The shallow water approximation also eliminates all terms that contain the meridional component $f_\theta = 2\Omega\cos\theta$ of the planetary vorticity vector (traditional approximation). The traditional approximation must be introduced with the hydrostatic approximation to maintain a representation of the Coriolis force that is consistent with the energy and angular momentum balance [*Phillips*, 1966]. The linear shallow water equations have separable solutions. The scaling arguments leading to (F1) are straightforward only for the off-equatorial case and require more care for equatorial regions [e.g., *Kamenkovich and Kulakov*, 1977; *Gill*, 1982]. In (F1a) and (F1b) the terms on the left-hand side scale as R̃o, Ro, Ro, Ro, Roε, and 1, respectively, and the right-hand side scales as 1. (R̃o is the temporal Rossby number.)

Beta Plane Equations

The beta plane equations are obtained by introducing a background stratification into the shallow water equations, expanding them around a reference latitude θ_0 with respect to $\varepsilon \sim \theta - \theta_0$, and keeping terms up to first order in ε. They are

$$\frac{\partial u}{\partial t} + u\left(1 + \frac{y}{r_0}\tan\theta_0\right)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$-\frac{uv}{r_0}\tan\theta_0 - f_0 v - \beta_0 y v$$

$$= -\frac{1}{\rho_*}\left(1 + \frac{y}{r_0}\tan\theta_0\right)\frac{\partial \delta p}{\partial x} \qquad \text{(F2a)}$$

where the terms on the left-hand side scale as R̃o, Ro, Roε, Ro, Ro, Roε, 1, and ε, respectively, and the terms on the right-hand side scale as 1 and ε, respectively,

$$\frac{\partial v}{\partial t} + u\left(1 + \frac{y}{r_0}\tan\theta_0\right)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{u^2}{r_0}\tan\theta_0$$

$$+ f_0 u + \beta_0 y u = -\frac{1}{\rho_*}\frac{\partial \delta p}{\partial y} \qquad \text{(F2b)}$$

where the terms on the left-hand side scale as the respective terms in (F2a) and the right-hand side scales as 1,

$$0 = g\delta\rho + \frac{\partial \delta p}{\partial z} \qquad \text{(F2c)}$$

where the terms on the right-hand side scale as 1,

$$\left(1 + \frac{y}{r_0}\tan\theta_0\right)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v}{r_0}\tan\theta_0 + \frac{\partial w}{\partial z} = 0 \qquad \text{(F2d)}$$

where the terms on the left-hand side scale as 1, ε, 1, ε, and 1, respectively, and

$$\left\{\frac{\partial}{\partial t} + u\left(1 + \frac{y}{r_0}\tan\theta_0\right)\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial t}\right\}\delta\rho$$

$$= wN^2\frac{g}{\rho_*} \qquad \text{(F2e)}$$

where the terms on the left-hand side scale as S, S, $S\varepsilon$, S, and S, respectively, and the right-hand side scales as 1. These beta plane equations are seldom used, except at the equator where all terms proportional to $\tan\theta_0$ vanish and where the beta term becomes the only manifestation of Earth's spherical geometry.

The *f* Plane Equations

The *f* plane equations are obtained by neglecting all terms of $O(\varepsilon)$ in (F2) and take the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - f_0 v = -\frac{1}{\rho_*}\frac{\partial \delta p}{\partial x} \qquad \text{(F3a)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + f_0 u = -\frac{1}{\rho_*}\frac{\partial \delta p}{\partial y} \qquad \text{(F3b)}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_*}\frac{\partial \delta \mathrm{p}}{\partial \mathrm{t}} - \delta\rho \mathrm{g} \qquad \text{(F3c)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(F3d)}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\delta\rho = wN^2\frac{g}{\rho_*} \qquad \text{(F3e)}$$

Here we have reinstated the acceleration terms in the vertical momentum balance, that is, suppressed the hydrostatic approximation but not the traditional approximation. The *f* plane equations describe motions in a stratified fluid that rotates uniformly with angular velocity $f_0/2$. If R̃o or Ro ≫ 1 the Coriolis accelerations can be neglected and the equations describe motions in a nonrotating stratified fluid.

APPENDIX G:
VARIATIONAL PRINCIPLE FOR STRATIFIED FLUIDS

Here we derive a variational principle for incompressible stratified Boussinesq fluids on an *f* plane, both in its Lagrangian and Eulerian forms.

Lagrangian Form

The starting point is the Lagrangian density (equation (59)) for incompressible fluids. When the equilibrium position **r** is used as a fluid label, the variational principle takes the form

$$\delta \int d\tau \iiint d^3r[\rho_0(\mathbf{r})(\tfrac{1}{2}\,\mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{U}) - \rho_0(\mathbf{r})\Phi + p(J-1)] = 0 \qquad \text{(G1)}$$

where $J = \partial(\mathbf{x})/\partial(\mathbf{r})$. The f plane approximation is accomplished by setting

$$\mathbf{U} = \mathbf{\Omega} \times \mathbf{x} = \frac{f_0}{2}\,\mathbf{e}_z \times \mathbf{x} \qquad \text{(G2)}$$

where $\mathbf{e}_z$ is the vertical unit vector. Stratification is introduced by assuming

$$\rho_0(\mathbf{r}) = \tilde{\rho}(r_3) \qquad \text{(G3)}$$

where $\tilde{\rho}(r_3)$ is the background density field. The Boussinesq approximation is accomplished by replacing $\rho_0(\mathbf{r})$ in front of the specific kinetic energy term by the constant reference density ρ_*.

The potential energy term can be rewritten as

$$\begin{aligned} \rho_0(\mathbf{r})\Phi &= \tilde{\rho}(r_3)gz \\ &= \tilde{\rho}(r_3)gr_3 + \tilde{\rho}(r_3)g(z-r_3) \end{aligned} \qquad \text{(G4)}$$

where $\Phi = gz$ with gravitational acceleration g has been used. The first term is the potential energy of the background stratification and the refore an irrelevant constant. The pressure term in (G1) can be rewritten as

$$\begin{aligned} p(J-1) &= (\tilde{p}(z) + \delta p)(J-1) \\ &= -[\tilde{p}(z) - \tilde{p}(r_3)] + \tilde{p}(z)J - \tilde{p}(r_3) \\ &\quad + \delta p(J-1) \end{aligned} \qquad \text{(G5)}$$

where again the second and third terms represent irrelevant constants. When all these expressions are substituted into the variational principle, it takes the form

$$\delta \int d\tau \int d^3r\,[\rho_*(\tfrac{1}{2}\,\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \dot{\mathbf{x}}\cdot\mathbf{U}) - T + \delta p(J-1)] = 0 \qquad \text{(G6)}$$

with the potential energy density

$$\begin{aligned} T &= \tilde{\rho}(r_3)g(z-r_3) - [\tilde{p}(z) - \tilde{p}(r_3)] \\ &= g\int_{r_3}^{z} dz'\,[\tilde{\rho}(r_3) - \tilde{\rho}(z')] \end{aligned} \qquad \text{(G7)}$$

It is the work done by the buoyancy or the Archimedean force when the fluid particle moves from its equilibrium position r_3 to its actual position z. When expanded with respect to the displacement $\zeta = z - r_3$, the potential energy density becomes

$$T = \rho_* N^2\,\tfrac{1}{2}\,\zeta^2 + \cdots \qquad \text{(G8)}$$

and is quadratic in ζ to lowest order. Note that the potential energy is only defined up to a constant (which we have set to zero for the equilibrium state) and that the potential energy density is only defined up to a nondivergent part. A further ambiguity arises since the potential energy of a fluid particle can be allocated to its equilibrium or actual position. Expression (G7) is a particular choice and implies that the Lagrange multiplier is δp. Other choices are possible and lead to different identification of the Lagrange multiplier.

Variations $\delta\mathbf{x}$ yield the momentum

$$\rho_*(\ddot{\mathbf{x}} + f_0\mathbf{e}_z \times \dot{\mathbf{x}}) = -\nabla\delta p - g[\tilde{\rho}(r_3) - \tilde{\rho}(z)]\mathbf{e}_z \qquad \text{(G9)}$$

and variations δp the incompressibility constraint $J = 1$.

Eulerian Form

For incompressible fluids the Eulerian variational principle (equation (E12)) in the **C** representation takes the form

$$\delta \int dt \iiint d^3x\,\rho_0(\mathbf{r})\left(\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + \Phi + C_2\frac{\partial r_2}{\partial t} + C_3\frac{\partial r_3}{\partial t} + \frac{\partial\varphi}{\partial t}\right) = 0 \qquad \text{(G10)}$$

where C_2, C_3, r_2, r_3, and φ are to be varied. Making the same approximations as before, one arrives at

$$\delta \int dt \iiint d^3x\left[\rho_*\left(\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + C_2\frac{\partial r_2}{\partial t} + C_3\frac{\partial r_3}{\partial t} + \frac{\partial\varphi}{\partial t}\right) + T\right] = 0 \qquad \text{(G11)}$$

where

$$\mathbf{u} = C_2\nabla r_2 + C_3\nabla r_3 + \nabla\varphi - \frac{f_0}{2}\,\mathbf{e}_z \times \mathbf{x} \qquad \text{(G12)}$$

Variations δC_2, δC_3, δr_2, δr_3, and $\delta\varphi$ yield the equations of motion (152). The pressure is identified as

$$\begin{aligned} \delta p &= p - \tilde{p}(z) \\ &= -\rho_*\left[\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + C_2\frac{\partial r_2}{\partial t} + C_3\frac{\partial r_3}{\partial t} + \frac{\partial\varphi}{\partial t} + \frac{T}{\rho_*}\right] \end{aligned} \qquad \text{(G13)}$$

An equivalent variational principle for a set of canonical conjugate variables was "derived" by *Henyey* [1983] using a "motivated recipe."

NOTATION

b	thermobaric coefficient.
$\mathbf{B}$	pseudomomentum vector.
c	concentration, sound speed.

$\mathbf{C}$	modified pseudomomentum vector.
e	specific internal energy.
$\mathbf{e}$	unit vector in physical space.
$\tilde{\mathbf{e}}$	unit vector in label space.
f	Coriolis parameter.
$\mathbf{F}$	flux vector.
H	vertical length scale.
J	Jacobian.
K	Jacobian.
L	horizontal length scale.
M	Jacobian.
$\mathbf{n}$	normal vector.
N	Brunt-Väisälä frequency.
p	pressure.
$\mathbf{P}$	eigenvector, momentum vector.
q	potential vorticity.
r	radial distance.
r_0	mean radius of Earth.
$\mathbf{r}$	initial or equilibrium position.
Ro	Rossby number.
$\mathbf{s}$	label vector.
S	salinity, vertical strain parameter.
t	time (in Eulerian description).
T	temperature.
$\mathbf{u}$	fluid velocity.
U	horizontal velocity scale.
$\mathbf{U}$	velocity of rotating frame.
u	zonal velocity component.
v	meridional velocity component, specific volume.
w	vertical velocity component.
W	vertical velocity scale.
$\mathbf{x}$	position vector.
D/Dt	material derivative.
$\boldsymbol{\nabla}$	gradient operator in physical space.
$\boldsymbol{\nabla}_h$	horizontal gradient operator.
$\tilde{\boldsymbol{\nabla}}$	gradient operator in label space.
Δ	Laplacian.
Δ_h	horizontal Laplacian.
$\mathcal{D}$	displacement gradient tensor.
$\mathcal{G}$	displacement gradient tensor.
$\mathcal{H}$	system matrix.
$\mathcal{L}$	Lagrangian density.
α	thermal expansion parameter.
β	beta parameter, haline contraction parameter.
Γ	adiabatic temperature gradient, modified Bernoulli function.
δ	aspect ratio.
ε	geometry parameter.
ζ	vertical displacement.
η	specific entropy.
θ	latitude, potential temperature.
κ	adiabatic compressibility.
μ	chemical potential.
ξ	surface elevation.
$\boldsymbol{\Pi}$	potential vorticity vector.
ρ	density.
ρ_{pot}	potential density.
τ	time (in Lagrangian description).
Φ	geopotential.
φ	longitude, velocity potential.
ψ	tracer, stream function.
$\boldsymbol{\Omega}$	rotation rate.
$\boldsymbol{\omega}$	relative vorticity.

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